Free spectrahedra and compatibility of quantum measurements

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Talk outline

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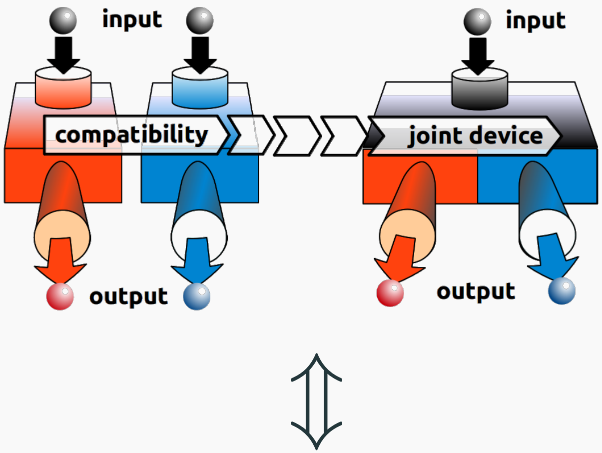
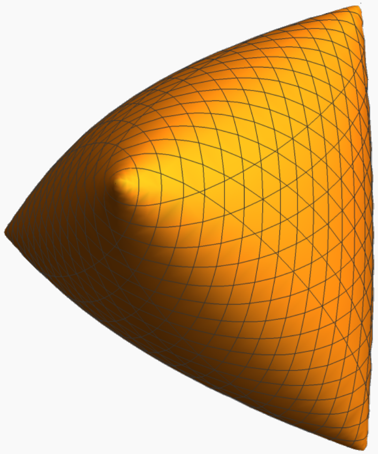




Incompatibility in QM

Measurements in Quantum Mechanics

- Quantum states $\rightsquigarrow$ density matrices: $\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$
- Quantum measurements $\rightsquigarrow$ give probabilities of obtaining outcomes

- Mathematically, measurements are modeled by POVMs

$$A_1, \ldots, A_k \in M_d(\mathbb{C})^{\mathrm{sa}}, \quad A_i \geq 0, \quad \sum_{i=1}^{k} A_i = I_d$$

- Born's rule: $\mathbb{P}[\text{outcome } i \text{ is observed}] = \operatorname{Tr}[\rho A_i]$

- Projective measurements: the A_i's are projections. Every POVM can be dilated to a projective measurement (Naimark's theorem)
- Trivial measurements: $A_i = p_i I_d$, for a probability vector p. Outcome probabilities do not depend on the input state
- Dichotomic measurements (or YES/NO POVMs): $k = 2$ outcomes, $A = (E, I_d - E)$ for an operator $0 \leq E \leq I_d$, called a quatum effect

Compatibility of measurements

- The position and momentum of a particle cannot be simultaneously measured $\rightsquigarrow$ Heisenberg's uncertainty relation: $\Delta_x \Delta_p \geq \hbar/2$

Definition

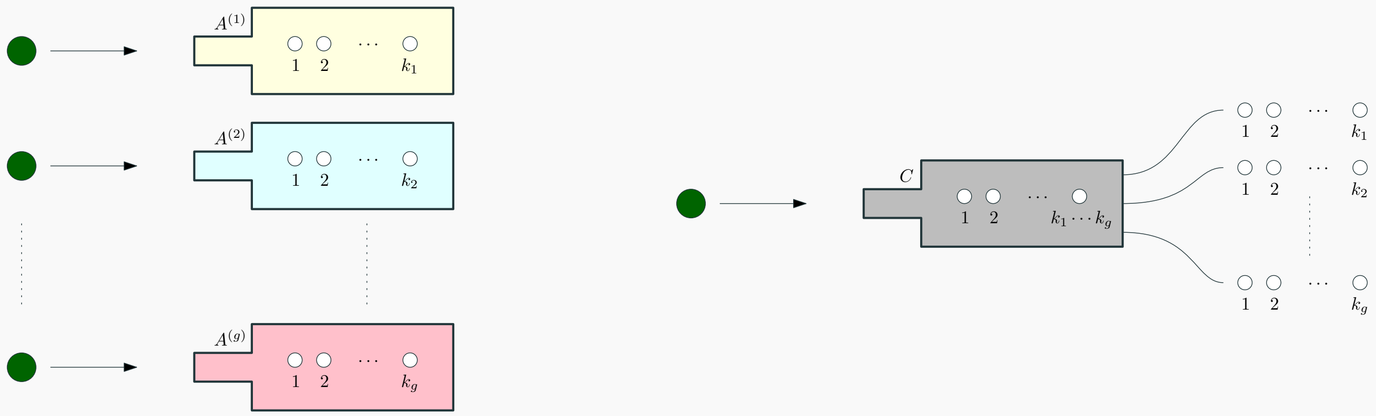
Two POVMs, $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_l)$, are called compatible if there exists a third POVM $C = (C_{ij})_{i \in [k], j \in [l]}$ such that

$$\forall i \in [k], \quad A_i = \sum_{j=1}^{l} C_{ij} \qquad \text{and} \qquad \forall j \in [l], \quad B_j = \sum_{i=1}^{k} C_{ij}$$

Similarly, a g-tuple of POVMs $A^{(1)}, \ldots, A^{(g)}$, having respectively $k_1, \ldots k_g$ outcomes, are said to be compatible if there exists a joint POVM C, with outcome set $[k_1] \times \cdots \times [k_g]$, such that

$$\forall x \in [g], \ , \forall i \in [k_x], \qquad A_i^{(x)} = \sum_{j \in [k_1] \times \cdots \times [k_g] \, : \, j(x) = i} C_j$$

What does it mean?



- Compatible measurements can be simulated by a single joint measurement, by classically post-processing its outputs
- Compatibility appears also in the setting of quantum channels [HMZ16]

Examples

- Trivial POVMs $A = (p_i I_d)$ and $B = (q_j I_d)$ are compatible

$$C_{ij} := p_i q_j I_d$$

- Commuting POVMs $[A_i, B_j] = 0$ are compatible

$$C_{ij} := A_i B_j = A_i^{1/2} B_j A_i^{1/2} = B_j^{1/2} A_i B_j^{1/2} \geq 0$$

- The following POVMs are not compatible

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

→ note that A_i, B_j are rank-1 projections

→ assume $(A_i), (B_j)$ compatible $\Rightarrow \exists\, C = (C_{ij})$ such that

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{matrix} \to A_1 \\ \to A_2 \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow \\ B_1 & B_2 \end{matrix}$$

→ $A_1 = C_{11} + C_{12}$ and $C_{ij} \geq 0 \Rightarrow C_{11} \sim A_1$

→ $B_1 = C_{11} + C_{21}$ and $C_{ij} \geq 0 \Rightarrow C_{11} \sim B_1$

$\Rightarrow C_{11} = 0$

→ similarly, $C_{ij} = 0 \;\forall ij \rightsquigarrow$ impossible since $\sum_{ij} C_{ij} = I$

Noisy POVMs

- POVMs can be made compatible by adding noise, i.e. mixing in trivial POVMs
- Example: dichotomic POVMs and white noise, $s \in [0,1]$

$$(E, I - E) \mapsto s(E, I - E) + (1-s)(\frac{I}{2}, \frac{I}{2}) \qquad \text{or} \qquad E \mapsto sE + (1-s)\frac{I}{2}$$

- Taking $s = 1/2$ suffices to render any pair of dichotomic POVMs compatible [DFK19] : take E, F two effects

Then, $C_{ij} := \frac{1}{4}E_i + \frac{1}{4}F_j$ is a joint POVM for $\frac{1}{2}E + \frac{1}{2}\frac{I}{2}$ and $\frac{1}{2}F + \frac{1}{2}\frac{I}{2}$

$\rightsquigarrow$ from now on, we focus on dichotomic (YES/NO) POVMs

Definition

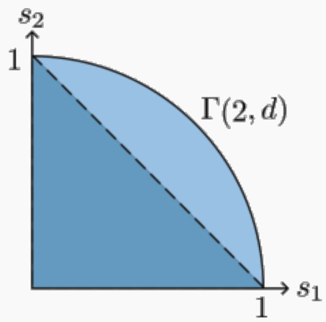
The compatibility region for g measurements on $\mathbb{C}^d$ is the set

$$\Gamma(g,d) := \{s \in [0,1]^g \ : \ \text{for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}), \text{ the noisy versions } s_i E_i + (1-s_i) I_d/2 \text{ are compatible}\}$$

Compatibility region

$$\Gamma(g,d) := \{s \in [0,1]^g \ : \text{ for all quantum effects } E_1,\ldots,E_g \in \mathcal{M}_d(\mathbb{C}), \text{ the noisy versions } s_iE_i + (1-s_i)I_d/2 \text{ are compatible}\}$$

- The set $\Gamma(g,d)$ is convex
- For all $i \in [g]$, $e_i \in \Gamma(g,d)$: every measurement is compatible with $g-1$ trivial measurements
- For $d \geq 2$, $(1,1,\ldots,1) \notin \Gamma(g,d)$: there exist incompatible measurements
- For all $d \geq 2$, $\Gamma(2,d)$ is a quarter-circle [BH08, BHSS13]
- Generally speaking, the set $\Gamma(g,d)$ tells us how robust (to noise) is the incompatibility of g dichotomic measurements on $\mathbb{C}^d$



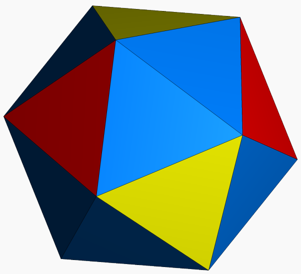
GOAL: compute the set $\Gamma(g,d)$ for all g,d

Free spectrahedra

Free spectrahedra

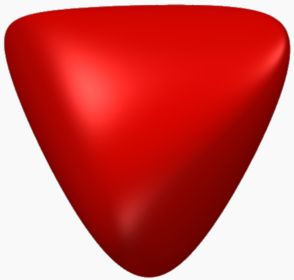
- A polyhedron is defined as the intersection of half-spaces

$$\{x \in \mathbb{R}^g \;:\; \langle h_i, x\rangle \leq 1, \quad \forall i \in [k]\}$$



- A spectrahedron is given by PSD constraints: for $A = (A_1, \ldots, A_g) \in (\mathcal{M}_d^{sa})^g$

$$\mathcal{D}_A(1) := \{x \in \mathbb{R}^g \;:\; \sum_{i=1}^{g} x_i A_i \leq I_d\}$$



- Example: $\mathcal{D}_{(\sigma_X, \sigma_Y, \sigma_Z)} =$ $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ ⇝ unit ball of ℓ^2

- A free spectrahedron is the matricization of a spectrahedron [Vin14]

$$\mathcal{D}_A := \bigsqcup_{n=1}^{\infty} \mathcal{D}_A(n) \quad \text{with} \quad \mathcal{D}_A(n) := \{X \in (\mathcal{M}_n^{sa})^g \;:\; \sum_{i=1}^{g} X_i \otimes A_i \leq I_{nd}\}$$

Examples: the cube and the diamond

The matrix cube is the free spectrahedron defined by

$$\mathcal{D}_{\square,g} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \|X_i\| \leq 1, \quad \forall i \in [g]\}$$

- At level one, $\mathcal{D}_{\square,g}(1)$ is the unit ball of the ℓ^∞ norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2g \times 2g$ diagonal matrices $\mathcal{D}_{\square,g} = \mathcal{D}_{K_1,\ldots,K_g}$, with $K_i = \mathrm{diag}(e_i) \oplus \mathrm{diag}(-e_i)$

The matrix diamond is the free spectrahedron defined by

$$\mathcal{D}_{\diamondsuit,g} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^{g} \varepsilon_i X_i \leq I_n, \quad \forall \varepsilon \in \{\pm 1\}^g\}$$

- At level one, $\mathcal{D}_{\diamondsuit,g}(1)$ is the unit ball of the ℓ^1 norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2^g \times 2^g$ diagonal matrices $\mathcal{D}_{\diamondsuit,g} = \mathcal{D}_{L_1,\ldots,L_g}$, with $L_i = I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2$

Spectrahedral inclusion

- Consider two free spectrahedra defined by g-tuples of matrices $(A_1, \ldots, A_g)$ and $(B_1, \ldots, B_g)$
- We say that $\mathcal{D}_A$ is contained in $\mathcal{D}_B$ and we write $\mathcal{D}_A \subseteq \mathcal{D}_B$ if, for all $n \geq 1$, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$
- Clearly, $\mathcal{D}_A \subseteq \mathcal{D}_B \implies \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$. For the converse implication to hold, one may need to shrink $\mathcal{D}_A$...

Definition

For a free spectrahedron $\mathcal{D}_A$, we define its set of inclusion constants as

$$\Delta_A(g, d) := \{s \in [0,1]^g \ : \text{for all } g\text{-tuples } B_1, \ldots, B_g \in \mathcal{M}_d(\mathbb{C})^{\mathrm{sa}}, \\ \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s.\mathcal{D}_A \subseteq \mathcal{D}_B\}$$

- The inclusion constants for the matrix cube play an important role in combinatorial optimization [HKMS19]
- We shall be concerned with the inclusion set for the matrix diamond, which we denote by $\Delta(g, d)$

Main results

Compatibility in QM $\iff$ matrix diamond inclusion

To a g-tuple of selfadjoint matrices $E \in (\mathcal{M}_d^{sa})^g$, we associate the free spectrahedron defined by the matrices $2E_i - I_d$:

$$\mathcal{D}_{2E-I} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^{g} X_i \otimes (2E_i - I_d) \leq I_{nd}\}$$

Theorem ([BN18])

Let $E \in (\mathcal{M}_d^{sa})^g$ be g-tuple of selfadjoint matrices. Then:

- *The matrices E are quantum effects $\iff \mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$*
- *The matrices E are compatible quantum effects $\iff \mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_{2E-I}$*

At the intermediate levels $1 \leq n \leq d$, $\mathcal{D}_{\diamond,g}(n) \subseteq \mathcal{D}_{2E-I}(n)$ iff for all isometries $V : \mathbb{C}^n \to \mathbb{C}^d$, the compressed effects V^*E_iV are compatible.

Moreover, the compatibility region is equal to the set of inclusion constants of the matrix diamond: $\forall g, d$, $\Gamma(g, d) = \Delta(g, d)$.

The same results hold in the general (non-dichotomic) setting. One has to replace the matrix diamond by the matrix jewel $\mathcal{D}_{\text{jewel},\mathbf{k}}$ [BN20]

Consequences

Many things are known about the matrix diamond

- For all g, d, $\frac{1}{2d}(1, 1, \ldots, 1) \in \Delta(g, d)$ [HKMS19]
- For all g, d, $\mathrm{QC}_g := \{s \in [0, 1]^g : \sum_i s_i^2 \leq 1\} \subseteq \Delta(g, d)$ [PSS18]

Many things are known about (in-)compatibility

- Many small g, d cases completely solved
- Approximate quanutum cloning [Kay16] $\implies$ compatibility

$$\mathrm{Clone}(g, d) := \{s \in [0, 1]^g : \exists \text{ quantum channel } \Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g} \text{ s.t.}$$
$$\forall i \in [g], \quad \Phi_i(X) = s_i X + (1 - s_i)\frac{\operatorname{Tr} X}{d}\}$$

Theorem ([BN18])

For all g and $d \geq 2^{\lceil (g-1)/2 \rceil}$, $\Gamma(g, d) = \Delta(g, d) = \mathrm{QC}_g$

Proof ideas

Inclusion of spectrahedra and (completely) positive maps

Theorem ([HKM13])

Let $A \in (\mathcal{M}_D^{sa}(\mathbb{C}))^g$, $B \in (\mathcal{M}_d^{sa}(\mathbb{C}))^g$ such that $\mathcal{D}_A(1)$ is bounded. Then, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ iff the unital linear map

$$\Phi : \mathrm{span}\{I, A_1, \ldots, A_g\} \to \mathcal{M}_d^{sa}(\mathbb{C})$$
$$A_i \mapsto B_i$$

is n-positive.

Sketch of the proof of the main theorem:

- Level 1: the extremal points of $\mathcal{D}_{\diamondsuit,g}(1)$ are $\pm e_i$
- The inclusion $\mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$ holds iff the unital map $\Phi : I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2 \mapsto 2E_i - I_d$ is CP
- Arveson's extension theorem [Pau02, Theorem 6.2]: Φ has a (completely) positive extension $\tilde{\Phi}$ to $\mathbb{R}^{2^g}$
- $C_\varepsilon := \tilde{\Phi}(\varepsilon)$ is a joint POVM for the E_i's, where $\{\varepsilon\}$ is a basis of $\mathbb{R}^{2^g}$

Maximally incompatible quantum effects

Lemma ([New32, Hru16])

For $d = 2^k$, there exist $2k+1$ anti-commuting, self-adjoint, unitary matrices $F_1, \ldots, F_{2k+1} \in \mathcal{U}_d$. Moreover, 2^k is the smallest dimension where such a $(2k+1)$-tuple exists.

- For $k = 0$, take $F_1^{(0)} := [1]$
- For $k \geq 1$, define $F_i^{(k+1)} = \sigma_X \otimes F_i^{(k)}$ $\forall i \in [2k+1]$ and $F_{2k+2}^{(k+1)} = \sigma_Y \otimes I_{2^k}$, $F_{2k+3}^{(k+1)} = \sigma_Z \otimes I_{2^k}$
- These matrices satisfy, for all $x \in \mathbb{R}_+^g$, $\left\| \sum_{i=1}^g x_i F_i \right\|_\infty = \|x\|_2$, and $\left\| \sum_{i=1}^g x_i \bar{F}_i \otimes F_i \right\|_\infty = \|x\|_1$
- For d large enough, the maximally incompatible g-tuple of quantum effects in $\mathcal{M}_d$ is given by $E_i = (F_i + I_d)/2$

The take-home slide

Measurement compatibility in QM $\iff$ free spectrahedron inclusion

- A measurement in QM (POVM): $A_i \in \mathcal{M}_d(\mathbb{C})$, $A_i \geq 0$, $\sum_i A_i = I_d$
- Measurements $(A_i), (B_j)$ are compatible if there exists a joint measurement (C_{ij}) having marginals A, B: $A_i = \sum_j C_{ij}$ and $B_j = \sum_i C_{ij}$

- Free spectrahedra: $\mathcal{D}_A := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^g X_i \otimes A_i \leq I_{nd}\}$
- Matrix diamond: $\mathcal{D}_{\diamond,g} = \bigsqcup_{n=1}^{\infty} \{X \in \mathcal{M}_n^g : \sum_{i=1}^g \varepsilon_i X_i \leq I_n, \forall \varepsilon \in \{\pm 1\}^g\}$

Theorem

Let $E \in (\mathcal{M}_d^{sa})^g$ be g-tuple of selfadjoint matrices. Then:

$$E \text{ are quantum effects} \iff \mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$$

$$E \text{ are compatible quantum effects} \iff \mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_{2E-I}$$

For all g, d, compatibility region $\Gamma(g,d) = \Delta(g,d)$ inclusion constants.

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