Enumerating meanders — three perspectives

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Talk outline

Enumerating meanders

Meandric systems with many connected components

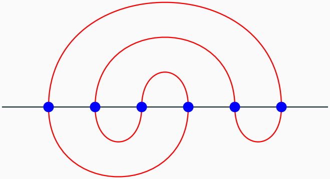
Shallow top meandric systems

Random matrix models

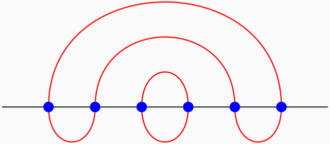
Enumerating meanders

Meanders

A meander of order n is a simple closed loop intersecting a horizontal line at $2n$ points.



A meandric system of order n is a non-intersecting set of simple closed loops intersecting a horizontal line at $2n$ points.

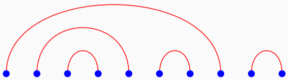


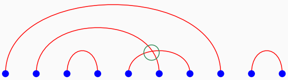
In other words, a meander is a meandric system with 1 connected component. The problem of enumerating meanders is a notoriously difficult open problem in combinatorics [DFGG97].

Non-crossing pairings and partitions

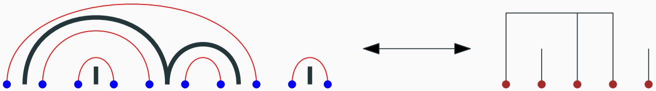
Meanders and meandric systems can be seen as the union of their top and their bottom parts. These are non-crossing pairings

$$\mathrm{NC}_2(2n) := \{\pi = \sqcup_{i=1}^{n}\{a_i, b_i\} \text{ partition of } [2n] \,:\, \nexists\, a_i < a_j < b_i < b_j\}.$$





Non-crossing pairings on $2n$ points are in bijection with non-crossing partitions on n points



Both sets are counted by the Catalan numbers $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$.

Formally,

$$\{\text{meandric systems of order } n\} = \{(\pi, \rho) \in \mathrm{NC}_2(2n)^2 \cong \mathrm{NC}(n)^2\}.$$

Number of connected components

Non-crossing partitions are in bijection with a subset of the symmetric group, the geodesic permutations

$$\sigma \in \mathcal{S}_{\mathrm{NC}}(n) \iff |\sigma| + |\sigma^{-1}\gamma| = |\gamma| = n - 1,$$

where $\gamma = (1\,2\,3\,\cdots\,n)$ and $|\cdot|$ denotes the length of a permutation

$$|\sigma| = \min\{k \,:\, \sigma = \tau_1\tau_2\cdots\tau_k \text{ with } \tau_i \text{ transpositions}\}.$$

For all permutations $\sigma \in \mathcal{S}(n)$, we have $|\sigma| = n - \#\sigma$, where $\#\sigma$ is the number of cycles of σ.

Proposition ([Nic16])

The number of connected components of the meandric system built out of two non-crossing partitions π, ρ is $\#(\pi^{-1}\rho)$.

We denote by $M_{n,r}$ the set of meandric systems of order n with r connected components $M_{n,r} = \{(\pi, \rho) \in \mathrm{NC}(n) \,:\, \#(\pi^{-1}\rho) = r\}$. In particular, $M_{n,1}$ is the set of meanders.

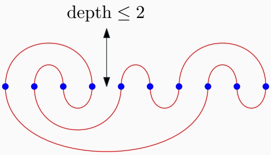
Number of meanders

Understanding the number of meanders $(|M_{n,1}|)_n$ (sequence A005315) is an important problem in combinatorics.

It is known that $\mathrm{Cat}_n \leq |M_{n,1}| \leq \mathrm{Cat}_n^2$.

Asymptotically, $|M_{n,1}| \sim Ca^n n^{-b}$, for constants C, a, b. It is known that $11.380 \leq a \leq 12.901$ [AP05]; numerically, $a \approx 12.269$. It is conjectured that $b = (29 + \sqrt{145})/12$ [DFGG00, DFGJ00].

Nica used free probability tools to study meanders in [Nic16]. Later, the notion of shallow top meanders was introduced: these are meanders (π, ρ) for which the top partition is an interval partition.



Proposition ([GNP20])

The number of shallow top meanders of order n is

$$M_{n,1}^{\mathrm{ST}} = \frac{1}{b} \sum_{k=1}^{n} \binom{n}{k-1} \binom{n+k-1}{n-k}.$$

Meandric systems with many connected components

Enumerating meandric systems

Recall that $M_{n,r}$ is the set of meandric systems on $2n$ points having r connected components

$$M_{n,r} = \{(\pi, \rho) \in \mathrm{NC}(n) \, : \, \#(\pi^{-1}\rho) = r\}.$$

$$|M_{n,1}| = |\{\text{meanders}\}| \qquad \text{(hard)}$$

$$\vdots$$

$$|M_{n,n-r}| = \quad ? \qquad r \text{ fixed}$$

$$|M_{n,n}| = \mathrm{Cat}_n \qquad (\text{easy}, \pi = \rho)$$

Generating series

Define the generating series

$$M(X,Y) := \sum_{n\geq 1}\sum_{r\geq 0} X^n Y^r |M_{n,r}|.$$

Compute

$$\begin{aligned}
M(X,Y) &= \sum_{n\geq 1} X^n \sum_{\pi,\rho\in NC(n)} Y^{|\pi^{-1}\rho|} \\
&= \sum_{n\geq 1} X^n \sum_{\omega\in NC(n)} \sum_{\substack{\pi,\rho\in NC(n)\\ \pi\vee\rho=\omega}} Y^{|\pi^{-1}\rho|} \\
&= \sum_{n\geq 1} X^n \sum_{\omega\in NC(n)} \prod_{b \text{ block of } \omega} \sum_{\substack{\pi,\rho\in NC(|b|)\\ \pi\vee\rho=1_{|b|}}} Y^{|\pi^{-1}\rho|} \\
&= \sum_{n\geq 1} X^n \sum_{\omega\in NC(n)} \prod_{b \text{ block of } \omega} \textsc{Function}(|b|, Y).
\end{aligned}$$

We recognize the moment - free cumulant formula from free probability theory [VDN92, NS06, MS17].

Moment - free cumulant transformations

Put

$$K_{n,r} := \{(\pi, \rho) \in \mathrm{NC}(n) \,:\, \pi \vee \rho = 1_n,\, |\pi^{-1}\rho| = r\}$$
$$K(X,Y) := \sum_{n \geq 1} \sum_{r \geq 0} X^n Y^r |K_{n,r}|.$$

The series M and K are related by the moment - free cumulant formula

$$M(X,Y) = K(X(1 + M(X,Y)), Y).$$

Using a similar reduction and a Kreweras complement, we can go deeper: if

$$I_{n,r} := \{(\pi, \rho) \in \mathrm{NC}(n) \,:\, \pi \wedge \rho = 0_n,\, \pi \vee \rho = 1_n,\, |\pi^{-1}\rho| = r\}$$
$$I(X,Y) := \sum_{n \geq 1} \sum_{r \geq 0} X^n Y^r |I_{n,r}|$$

then

$$K(X,Y) = I(X(1 + K(X,Y)), Y).$$

Moment - free cumulant transformations

Recall

$$M_{n,r} = \{(\pi,\rho) \in \mathrm{NC}(n) \,:\, |\pi^{-1}\rho| = r\}$$
$$K_{n,r} = \{(\pi,\rho) \in \mathrm{NC}(n) \,:\, \pi \vee \rho = 1_n,\, |\pi^{-1}\rho| = r\}$$
$$I_{n,r} = \{(\pi,\rho) \in \mathrm{NC}(n) \,:\, \pi \wedge \rho = 0_n,\, \pi \vee \rho = 1_n,\, |\pi^{-1}\rho| = r\}$$

and let M, K, I the respective generating series.

If $\mathcal{F}$ is the operation transforming free cumulant generating series into moment generating series, we conclude

$$I \xmapsto{\mathcal{F}_X} K \xmapsto{\mathcal{F}_X} M$$

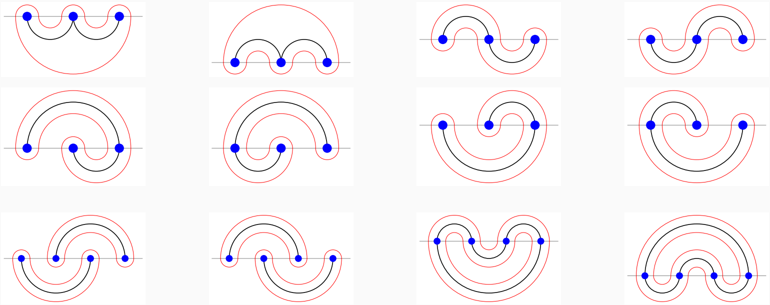
The sets $I_{n,r}$ should be easier to enumerate...

Key technical lemma

Lemma

For fixed r, the series I has finite support in n. More precisely, $I_{n,r} = \emptyset$, unless $r+1 \leq n \leq 2r + \mathbf{1}_{r=0}$.

For $r = 1$, $I_{2,1} = \{(\downarrow\ \downarrow, \sqcap), (\sqcap, \downarrow\ \downarrow)\}$, and all the other $I_{n,1}$ are empty.



All meanders in $I_{n,r=2}$. We have $[Y^2]I(X,Y) = 8X^3 + 4X^4$.

The main theorem

Recall that $|M_{n,s}|$ is the number of meandric systems of order n with s connected components.

Theorem ([FN19])

For any fixed $r \geq 1$ there exists a polynomial $\tilde{P}_r$ of degree at most $3r-3$ such that the generating function of the number of meandric systems of order n with $n-r$ connected components

$$F_r(t) = \sum_{n=r+1}^{\infty} |M_{n,n-r}| t^n,$$

after the change of variables $t = w/(1+w)^2$, reads

$$F_r(t) = \frac{w^{r+1}(1+w)}{(1-w)^{2r-1}} \tilde{P}_r(w).$$

Exact results and asymptotics

With the help of a computer, we can enumerate $I_{n,r}$ for $1 \leq r \leq 6$ (we just have to look at $\mathrm{NC}(\leq 12)$ to do this) to find

$$\tilde{P}_1(w) = 2$$
$$\tilde{P}_2(w) = 4w^3 - 12w^2 + 4w + 8$$
$$\tilde{P}_3(w) = 18w^6 - 92w^5 + 134w^4 + 8w^3 - 146w^2 + 52w + 42$$
$$\vdots$$

Corollary

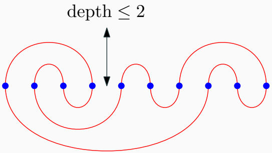
For any fixed $r \geq 1$, assuming that $\tilde{P}_r(1) \neq 0$ (this holds at least for $1 \leq r \leq 6$), the number of meandric systems of order n having $n - r$ connected components has the following asymptotic behavior:

$$|M_{n,n-r}| \sim \frac{\tilde{P}_r(1)}{2^{2r-2}\Gamma((2r-1)/2)} 4^n n^{(2r-3)/2}.$$

Shallow top meandric systems

Shallow top meandric systems

Recall that shallow top meanders are meanders (π, ρ) with the property that the top partition ρ is an interval partition: its blocks are made out of consecutive integers.



We can replace one free transform with a boolean transform

$$I \xmapsto{\mathcal{F}_X} K \xmapsto{\mathcal{B}_X} M.$$

Theorem ([FN21])

The generating function for shallow top meandric systems is given by

$$M^{\mathrm{ST}}(X, Y, A, B) = \sum_{n=1}^{\infty} X^n \sum_{\substack{\pi \in \mathrm{Int}(n) \\ \rho \in \mathrm{NC}(n)}} Y^{|\pi^{-1}\rho|} A^{|\pi|} B^{|\rho|} = \frac{K(X, Y, A, B)}{1 - K(X, Y, A, B)}.$$

where $K(X) = h(X(1 + \hat{g}(X)))$, *with* $\hat{g} = \mathcal{F}(g)$, *and*

$$g(X) = \sum_{n=1}^{\infty} g_n X^n \text{ where } g_n = BY\left[(1 + AY)^n + (AY)^n (Y^{-2} - 1)\right],$$

$$h(X) = \sum_{n=1}^{\infty} h_n X^n \text{ where } h_n = (AY)^{n-1}.$$

Random matrix models

Previous constructions

- Recall the meander polynomial

$$m_n(Y) = \sum_{\alpha,\beta \in NC(n)} Y^{\#(\alpha^{-1}\beta)}.$$

- di Francesco showed [DFGG97] that it can be related to the moments of a random matrix model built out of tensor products of GUE matrices, for integer $Y = k$:

$$m_n(k) = \lim_{d\to\infty} \mathbb{E}\frac{1}{d^2} \operatorname{Tr} \left(\sum_{i=1}^{k} \frac{B_i \otimes \bar{B}_i}{d} \right)^{2n},$$

where $B_1, \ldots, B_k \in \mathcal{M}_d(\mathbb{C})$ are i.i.d. GUE matrices.

- Fukuda and Śniady relate [FŚ13] the meander polynomial to the partial transposition of Wishart matrices: if W is a Wishart matrix of parameters (d^2, k), then

$$m_n(k) = \lim_{d\to\infty} \mathbb{E}\frac{1}{d^2} \operatorname{Tr} \left(\frac{W^\Gamma}{d} \right)^{2n},$$

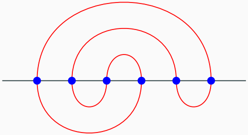
where $W^\Gamma = [\mathrm{id} \otimes \mathrm{transp}](W)$ is the partial transposition of W.

New models

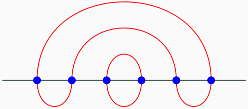
- We show [FN21] that it is related to tensor products of independent random completely positive maps, applied to a maximally entangled state.
- Let $G, H \in \mathcal{M}_{d^2 \times k}(\mathbb{C})$ be two independent Ginibre matrices. Consider the random CP maps $\Phi_{G,H} : \mathcal{M}_k(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$, where
$$\Phi_A(X) = [\mathrm{id} \otimes \mathrm{Tr}](AXA^*).$$
- Define $Z := [\Phi_G \otimes \Phi_H](\omega_k) \in \mathcal{M}_{d^2}(\mathbb{C})$, where ω_k is the max. ent quantum state $\omega_k = \sum_{i,j=1}^k e_i \otimes e_i \cdot (e_j \otimes e_j)^*$.
- Then, for all $n, k \geq 1$,
$$m_n(k) = \lim_{d \to \infty} \mathbb{E} \frac{1}{d^2} \mathrm{Tr} \left(\frac{Z}{d^2} \right)^n.$$
- For shallow top meanders, replace one of the CP maps with a depolarizing channel $\Psi(X) = X + \mathrm{Tr}(X)I$:
$$m_n^{\mathrm{ST}}(k) = \lim_{d \to \infty} \mathbb{E} \frac{1}{d} \mathrm{Tr}[(d^{-1} Z_0)(d^{-1} Z)^{n-1}],$$
where $Z := [\Phi_G \otimes \Psi](\omega_k)$ and $Z_0 := [\Phi_G \otimes \mathrm{id}](\omega_k)$ are $dk \times dk$ matrices.

Take home slide

A meander of order n is a simple closed loop intersecting a horizontal line at $2n$ points.



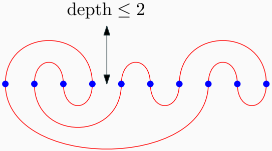
A meandric system of order n is a non-intersecting set of simple closed loops intersecting a horizontal line at $2n$ points.



GOAL: Enumerate meanders (asymptotically).

Result 1: Generating function for meandric systems with large number of connected components $n - r$, for small r.

Shallow top meanders are meanders (π, ρ) with the property that the top partition ρ is an interval partition.



Result 2: Explicit generating function for shallow top meandric systems.

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