An introduction to tensors

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Talk outline

Definition and graphical notation

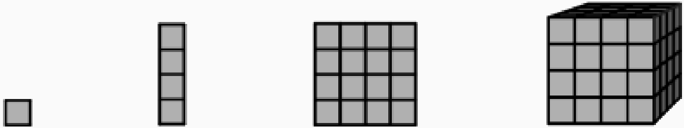
Tensor rank(s)

Tensor decompositions

Definition and graphical notation

What is a tensor?

- A scalar (0D object): $x \in \mathbb{C}$
- A vector (1D object): $v = (v_i)_{i\in[d]} \in \mathbb{C}^d$
- A matrix (2D object): $A = (A_{ij})_{i,j\in[d]} \in \mathcal{M}_d(\mathbb{C}) \cong \mathbb{C}^d \otimes \mathbb{C}^d$
- A tensor (3D object): $T = (T_{ijk})_{i,j,k\in[d]} \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$



by Anna Seigal

Definition

A *n*-mode tensor is an element $T \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$. It can be decomposed in the computational basis as

$$T = \sum_{i\in[d_1]\times\cdots\times[d_n]} T_{i_1 i_2 \cdots i_n} \underbrace{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}}_{=:|i_1 i_2 \cdots i_n\rangle} .$$

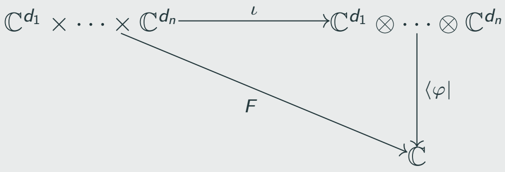
Abstract definition

Tensors are related to multilinear forms via the universal-property.

Proposition

Any $F : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \cdots \times \mathbb{C}^{d_n} \to \mathbb{C}$ *factors through* $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ *uniquely. There exists a unique linear form* $\langle \varphi | \in [\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}]^*$ *such that*

$$F(x_1, x_2, \ldots, x_n) = \langle \varphi, x_1 \otimes x_2 \otimes \cdots \otimes x_n \rangle, \qquad \forall x_i.$$



where $\iota(x_1, \ldots, x_n) = x_1 \otimes \cdots \otimes x_n$.

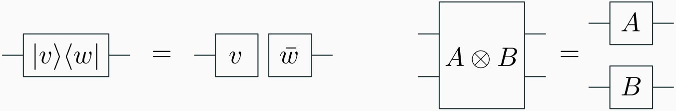
Graphical notation: tensor diagrams

A tensor diagram is a collection of boxes and wires.

❶ Boxes ⇝ tensors

- A box with n legs corresponds to a n-tensor. Each leg is associated to a vector space $\mathbb{C}^{d_i}$.
- Scalars are leg-less boxes, vectors have 1 leg, matrices have two legs, etc.

- The Kronecker product of tensors corresponds to juxtaposition.



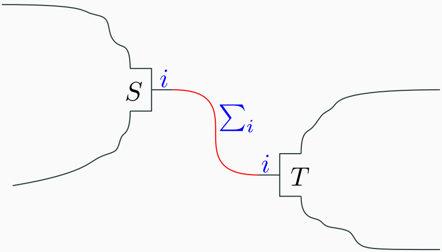
- Writing indices over a leg gives the corresponding coordinate.

Graphical notation: tensor diagrams

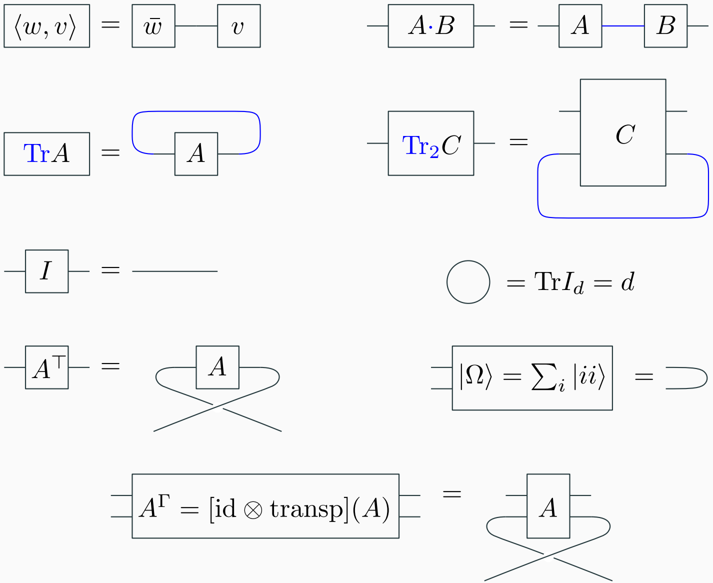
❷ Wires ⇝ tensor contractions

- A wire between two tensors corresponds to a sum over a common index. Only legs corresponding to the same vector space can be connected by a wire. Mathematically, a wire corresponds to the evaluation map:

$$\begin{aligned} \mathrm{ev} : V^* \times V &\to \mathbb{C} \\ (\alpha, v) &\mapsto \alpha(v). \end{aligned}$$



Operations on tensors



Tensor rank(s)

Tensor rank

A tensor of the form $x = x_1 \otimes x_2 \otimes \cdots \otimes x_n$ is called a simple (or product) tensor. In QIT, these are the separable pure states.

Definition

The rank of a tensor x is the minimum number of terms in a decomposition of x as a sum of simple tensors

$$R(x) := \min\{r : x = \sum_{i=1}^{r} x_1^{(i)} \otimes x_2^{(i)} \otimes \cdots \otimes x_n^{(i)}\}.$$

For matrices, this is the usual notion of rank:

$$R(A) = \min\{r : A = \sum_{i=1}^{r} |x_i\rangle\langle y_i|\}.$$

Examples for 3-qubit tensors:

- The GHZ state $|\mathrm{GHZ}\rangle = |000\rangle + |111\rangle$ has rank 2.
- The W state $|\mathrm{W}\rangle = |001\rangle + |010\rangle + |100\rangle$ has rank 3.

Border rank

In the case of matrices, the set of tensors of rank $\leq r$ is an algebraic variety (i.e. 0-set of some polynomials), hence closed.

For 3-tensors, this is no longer true: one can write a tensor of rank 3 as a limit of rank-2 tensors:

$$|\mathrm{W}\rangle = |001\rangle + |010\rangle + |100\rangle = \lim_{\varepsilon \to 0} \frac{(|0\rangle + \varepsilon |1\rangle)^{\otimes 3} - |000\rangle}{\varepsilon}.$$

Definition

Border rank of a tensor

$$\underline{R}(x) := \min\{r : x = \lim_{k \to \infty} x_k, \text{ with } R(x_k) = r \quad \forall k\}.$$

So $\underline{R}(|\mathrm{W}\rangle) = 2 < 3 = R(|\mathrm{W}\rangle)$.

Symmetric tensors

An element of $\vee^n(\mathbb{C}^d)$ is called a symmetric tensor. The tensors $|\mathrm{GHZ}\rangle$ and $|\mathrm{W}\rangle$ are symmetric.

Since $\mathrm{span}\{x^{\otimes n}\}_{x\in\mathbb{C}^d} = \vee^n(\mathbb{C}^d)$, one can define the symmetric rank

$$R_s = \min\{r \,:\, x = \sum_{i=1}^{r} x_i^{\otimes n}\}.$$

Clearly, $R_s \geq R$ for symmetric tensors. The inequality can be strict (Shitov's counterexample to Comon's conjecture from 2018, tensor of size 800^3, with $R \leq 903$ and $R_s \geq 904$).

Remarkably, symmetric tensors are in bijection with homogeneous polynomials in d variables of degree n

$$t \in \vee^n(\mathbb{C}^d) \leftrightarrow p(x_1, \dots x_d) = \langle t, (x_1, \dots, x_d)^{\otimes n}\rangle.$$

This correspondence has many interesting properties (GHZ-type tensors correspond to norms, partial traces correspond to derivatives, etc).

Tensor decompositions

The singular value decomposition

Any $d \times d$ matrix A can be decomposed as

$$A = \sum_{i=1}^{d} s_i |x_i\rangle\langle y_i|,$$

where $s_i \geq 0$ are the singular values and $\{x_i\}$, resp $\{y_i\}$ are orthonormal bases of $\mathbb{C}^d$. The number of non-zero singular values is the rank of A.

In QIT, this is know as the Schmidt decomposition

$$\mathbb{C}^d \otimes \mathbb{C}^d \ni |\varphi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |x_i\rangle \otimes |y_i\rangle \, .$$

If the vector $|\varphi\rangle$ is normalized, the Schmidt coefficients $\{\lambda_i\}_{i \in [d]}$ form a probability vector. The Schmidt coefficients contain most of the physically relevant information in the bipartite pure state $|\varphi\rangle$, such as the entropy of entanglement

$$E(|\varphi\rangle) = H(\lambda) = -\sum_{i=1}^{d} \lambda_i \log \lambda_i.$$

Tensor decompositions

Not all tensors ($n \geq 3$) can be decomposed as

$$x = \sum_{i=1}^{d} s_i x_i^{(1)} \otimes x_i^{(2)} \otimes \cdots \otimes x_i^{(n)},$$

with orthonormal $\{x_i^{(k)}\}_{i \in [d]}$ for all k. Naive dimension count:

$$\dim_{\mathbb{R}} (\mathbb{C}^d)^{\otimes n} = 2d^n \gg d + nd^2 - d.$$

Tensors admitting such a decomposition are called udeco. Their best rank-1 approximation is given by the term with the largest s_i.

Tucker and HOSVD decompositions: isometries + a core tensor

$$\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n} \ni T = (V_1 \otimes V_2 \otimes \cdots \otimes V_n) C,$$

with $C \in \mathbb{C}^{p_1} \otimes \cdots \otimes \mathbb{C}^{p_n}$ and $V_i : \mathbb{C}^{p_i} \to \mathbb{C}^{d_i}$ are isometries. For some applications, we want C to be as "diagonal" as possible.