Practice exercises

Exercise 1

The goal of this exercise is to discover the interesting fact that the spectrum of (large) random matrices has a universal behavior.

1. Let $N = 100$. Construct, in `python`, a $N \times N$ random matrix X with entries from $[0, 1]$
2. Symmetrize the matrix, in such a way that $X = X^\top$
3. Compute the eigenvalues of the symmetric matrix.
4. Plot a histogram of the eigenvalues. What do you see?
5. Repeat the experiment for $N = 10, 20, 50, 1000$. Interpret the results.

Exercise 2

In this exercise, we shall introduce and study the properties of the trace distance between two pure states.

Let $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ be two pure states on an arbitrary Hilbert space $H = \mathbb{C}^d$.

1. Let $\|\cdot\|_1$ denote the *trace norm*, which for a Hermitian operator A with eigenvalues (a_i) is defined by
$$\|A\|_1 := \sum_i |a_i|.$$
What is the trace norm of the identity operator on H? What is the trace norm of a positive semidefinite operator?

2. Show that
$$\frac{1}{2}\|\rho - \sigma\|_1 = \sqrt{1 - |\langle\psi|\phi\rangle|^2}.$$
Hint: Argue that the eigenvalues of $\rho - \sigma$ are of the form $(\lambda, -\lambda, 0, \ldots, 0)$ for some $\lambda \in \mathbb{R}$. Compute the two quantities in terms of λ.

3. Can you relate the latter to $|\langle\psi|\phi\rangle|^2$?

Exercise 3

Argue that the teleportation protocol can be mathematically understood in terms of the following *teleportation identity*.

1. Let $|\psi\rangle$ be a qubit pure state. Then

$$|\psi\rangle \otimes \left|\Omega^{00}\right\rangle = \frac{1}{2} \sum_{z,x \in \{0,1\}^2} |\Omega^{zx}\rangle \otimes X^x Z^z |\psi\rangle .$$

2. In the teleportation protocol discussed in the lectures, what is the state that Alice ends up with at the end of the protocol?
3. Could Alice have ended up with the unknown state $|\psi\rangle$ in one of her qubits?

   Hint: use the no-cloning theorem.