QUANTUM CHANNELS

① Pure and mixed quantum states

Recall

- pure quantum states

  $|\varphi\rangle \in \mathbb{C}^d \qquad \|\varphi\| = 1$

  → classical states (basis elements)

  $|0\rangle, |1\rangle \in \mathbb{C}^2$ qubit

  → non-classical states

  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

  $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

  superpositions of pure states

- mixed quantum states

  $\rho \in M_d(\mathbb{C}) \qquad \rho \geq 0 \qquad \mathrm{Tr}\,\rho = 1$

  positive semidefinite
  eigenvalues ≥ 0

  → pure states $\rho = |\varphi\rangle\langle\varphi|$

  → maximally mixed state

  $\rho = I/d$

  → qubits : Bloch ball

  $\rho = \frac{1}{2}(I + xX + yY + zZ)$

  $\vec{r} = (x, y, z) \qquad \|\vec{r}\| \leq 1$

  ↳ pure states → Bloch sphere $\|\vec{r}\| = 1$

Purification Any mixed quantum state can be purified into a bipartite pure state

A	A + B
mixed	pure
single system	bipartite system.

② The partial trace operation

Assume that Alice has a q. state ρ_A, and Bob has ρ_B. Then, the q state held jointly by Alice an Bob is

$$\rho_{AB} = \rho_A \otimes \rho_B$$

Definition If Alice and Bob have a q. syst ρ_{AB}, then Alice holds the quantum syst.

$$\rho_A = Tr_B \, \rho_{AB}$$

where the partial trace operation Tr_B is defined by

$$Tr_B (X \otimes Y) = [\underset{A}{id} \otimes \underset{B}{Tr}](X \otimes Y) = X \cdot Tr(Y)$$

__Examples__

- $\mathrm{Tr}_B\left(\sum_i X_i \otimes Y_i\right) = \sum_i X_i \cdot \mathrm{Tr}(Y_i)$

"linearity"

- $\mathrm{Tr}_B(\rho_A \otimes \rho_B) = \rho_A \cdot \underbrace{\mathrm{Tr}(\rho_B)}_{=1} = \rho_A$

"the partial trace operation is the inverse of the tensor product operation"

- $\rho_{AB} = \omega = |\Omega\rangle\langle\Omega|$ maximally entangled state

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$$

$$\omega = |\Omega\rangle \cdot \langle\Omega| = \left[\frac{1}{\sqrt{d}} \sum_i |ii\rangle\right] \cdot \left[\frac{1}{\sqrt{d}} \sum_j \langle jj|\right]$$

$$= \frac{1}{d} \sum_{i,j=1}^{d} \underbrace{|ii\rangle}_{|i\rangle \otimes |i\rangle}\langle jj| = \frac{1}{d} \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes |i\rangle\langle j|$$

So $\mathrm{Tr}_B(\omega) = \mathrm{Tr}_B\left(\frac{1}{d} \sum_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j|\right)$

$$= \frac{1}{d} \sum_{ij} \mathrm{Tr}_B\left(|i\rangle\langle j| \otimes |i\rangle\langle j|\right)$$

$$= \frac{1}{d} \sum_{ij} |i\rangle\langle j| \cdot \underbrace{\mathrm{Tr}\left(|i\rangle\langle j|\right)}_{= \langle j|i\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}}$$

$$= \frac{1}{d} \sum |i\rangle\langle i| = I_d/d$$

$$\mathrm{Tr}_B\left(|\Omega\rangle\langle\Omega|\right) = \frac{I}{d}$$

· partial trace as the adjoint of "$\otimes I_B$"

$$\mathrm{Tens}_B : M_{d_A} \xrightarrow{\otimes I_B} M_{d_A} \otimes M_{d_B}$$

$$X \longmapsto X \otimes I_B$$

↳ What is Tens_B^* ?

$$\forall X, Z : \langle \mathrm{Tens}_B X, Z \rangle = \langle X, \mathrm{Tens}_B^* Z \rangle$$

$$\Leftrightarrow \forall X, P, Q \quad \langle \mathrm{Tens}_B X, P \otimes Q \rangle = \langle X, \mathrm{Tens}_B^* (P \otimes Q) \rangle$$

$$\begin{aligned} &\langle \mathrm{Tens}_B X, P \otimes Q \rangle \\ &= \langle X \otimes I_B, P \otimes Q \rangle \\ &= \langle X, P \rangle \cdot \underbrace{\langle I_B, Q \rangle}_{\mathrm{Tr}\, Q} = \langle X, P \rangle \cdot \mathrm{Tr}\, Q \end{aligned}$$

$$\langle X, \mathrm{Tens}_B^* (P \otimes Q) \rangle = \langle X, P \rangle \cdot \mathrm{Tr}\, Q$$

$$\forall X \quad \langle X, \mathrm{Tens}_B^* (P \otimes Q) \rangle = \langle X, P \rangle \cdot \mathrm{Tr}\, Q = \langle X, P \cdot \mathrm{Tr}\, Q \rangle$$

$$\mathrm{Tens}_B^* (P \otimes Q) = P \cdot \mathrm{Tr}\, Q = \mathrm{Tr}_B (P \otimes Q)$$

3 Purification

Fact For any mixed quantum state ρ_A, there exist a pure state $|\varphi\rangle_{AB}$ such that

$$\rho_A = \mathrm{Tr}_B\, |\varphi\rangle\langle\varphi|_{AB}$$

The pure state $|\varphi\rangle_{AB}$ is called a purification of ρ_A

Example $|\Omega\rangle_{AB}$ is a purification of $\rho_A = I/d$

Proof Let ρ_A be an arbitrary mixed state.

ρ_A is PSD, hence self-adjoint, hence normal
$\rightsquigarrow$ use spectral decomposition

$$\rho_A = \sum \underbrace{\lambda_i}_{\text{eigenvalues}} \, |\underbrace{a_i}_{\text{eigenvectors}}\rangle\langle a_i|$$

$\rightarrow \rho_A$ is a q. state : $\lambda_i \geq 0$, $\sum \lambda_i = 1$

Idea : use the spectral decomposition of ρ_A to construct the Schmidt decomp. of $|\varphi\rangle_{AB}$

$$|\varphi_{AB}\rangle := \sum_i \sqrt{\lambda_i} \, |a_i\rangle \otimes |b_i\rangle$$

where $\{|b_1\rangle, \ldots, |b_d\rangle\}$ is an arbitrary orthonormal basis of $\mathbb{C}^d$.

↳ above, we have the Schmidt decomp of $|\varphi\rangle$

$\rightarrow$ $\lambda_i \geq 0$, $\sum \lambda_i = 1$

$\rightarrow$ $\{a_i\}$ o.n.b. of $\mathbb{C}^d$

$\rightarrow$ $\{b_i\}$ —— " ——

Let us check that $|\varphi\rangle_{AB}$ is a purification of ρ_A

$$\mathrm{Tr}_B |\varphi\rangle\langle\varphi| = \sum_{ij} \sqrt{\lambda_i \lambda_j} \, \mathrm{Tr}_B\left(|a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|\right)$$

$$= \sum_{ij} \sqrt{\lambda_i \lambda_j} \, |a_i\rangle\langle a_j| \cdot \underbrace{\mathrm{Tr}\left(|b_i\rangle\langle b_j|\right)}_{= 1 \text{ if } i=j \,;\, 0 \text{ if } i \neq j} = \rho_A .$$

④ Quantum channels. Definition and motivation

Recall Time evolution for closed (isolated, single) q. syst
$\rightarrow$ unitary operators

$$\rho' = U \rho U^*$$
$$|\varphi'\rangle = U \cdot |\varphi\rangle$$

What if we want to describe the time evolution of Alice's q. syst, which is in contact with another q. syst (Bob's)?

Definition A quantum channel is a linear map

$$\Phi : M_d \longrightarrow M_D \quad \text{satisfying}$$

- Φ is completely positive: CP
  $\forall n \quad \Phi \otimes id_n : M_d \otimes M_n \to M_D \otimes M_n$
  is positive
  $\uparrow$
  if $X \geq 0$ then $[\Phi \otimes id_n](X) \geq 0$
- Φ is trace-preserving TP
  $\forall X \quad Tr[\Phi(X)] = Tr[X]$

We say that a q. channel is a TPCP linear map

Remark A CP map is, in particular, positive:
$X \geq 0 \Rightarrow \Phi(X) \geq 0$. So
$\Phi(\text{density matrices}) \subseteq \{\text{density matrices}\}$

i.e. q. channels map q. states to q. states.

Examples

- closed unitary evolution

  $\Phi_U(X) = UXU^*$ is a q. channel whenever U is a unitary matrix

  ↳ TP: $\mathrm{Tr}(UXU^*) = \mathrm{Tr}(X\underbrace{U^*U}_{I}) = \mathrm{Tr}\,X$

  ↳ CP: $X \in M_d \otimes M_m \quad X \geq 0$

  $$(\Phi_U \otimes \mathrm{id}_m)(X) = (\Phi_U \otimes \Phi_{I_m})(X)$$
  $$= \Phi_{U \otimes I_m}(X)$$
  $$= (U \otimes I)(X)(U \otimes I)^* \geq 0$$

  since unitary conj. does not change eigs.

- depolarizing channel

  $$\Delta : M_d \longrightarrow M_D$$
  $$X \longmapsto (\mathrm{Tr}\,X) \cdot \frac{I_D}{D}$$

  ↳ TP: $\mathrm{Tr}(\Delta(X)) = (\mathrm{Tr}\,X) \cdot \underbrace{\frac{\mathrm{Tr}\,I}{D}}_{=1} = \mathrm{Tr}\,X$

$\hookrightarrow$ CP: $0 \leq X \in M_d^{(A)} \otimes M_m^{(B)}$

$$(\Delta \otimes id_m)(X) = \frac{I_D}{D} \otimes \underbrace{Tr_A(X)}_{\geq 0 \text{ since } X \geq 0} \geq 0$$

Partial trace operation in coordinates

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{bmatrix}$$

$$Z = \begin{bmatrix} z_{11} & \cdots & z_{1m} \\ \vdots & & \vdots \\ z_{m1} & \cdots & z_{mm} \end{bmatrix} \in M_m \otimes M_n$$

$$Z_{ij} \in M_n$$

$$Tr_m(Z) = z_{11} + z_{22} + \cdots + z_{mm} \in M_n$$

"sum over diagonal blocks"

$$Tr_n(Z) = \begin{bmatrix} Tr\, z_{11} & \cdots & Tr\, z_{1m} \\ \vdots & & \vdots \\ Tr\, z_{m1} & \cdots & Tr\, z_{mm} \end{bmatrix} \in M_m$$

"matrix of traces"

- dephasing channel

$$\phi : M_d \to M_d$$
$$X \mapsto \operatorname{diag}(X)$$

$d=2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

A non-example : the TRANSPOSITION map

$$\theta : M_d \longrightarrow M_d$$
$$X \longrightarrow X^T$$

- ϕ is linear, TP : $Tr(X^T) = Tr(X)$
- ϕ is positive : $X \geq 0 \Rightarrow X^T \geq 0$

X^T has the same spectrum as X

In particular, θ maps q. states to q. states

- however, θ is NOT completely positive

↳ consider $d = 2$, $n = 2$

$\sigma := (\theta_2 \otimes id_2)(\omega)$

maximally entangled state of 2 qubits

$d=2$ $\quad |\Omega_{d=2}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{array}{cc} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} \end{array}$

$$\omega = \frac{1}{2} \begin{array}{cc} \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} & \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] & \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} \end{array}$$

$$\sigma = [\theta \otimes id] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\sigma = [1] \oplus [1] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

In particular, σ has eigenvalues

$$\{1\} \cup \{1\} \cup \{-1, 1\} \Rightarrow \sigma \not\geq 0$$

$\Rightarrow$ θ_2 is not completely positive!

(in particular, θ is NOT a q. channel)

Examples for qubits

- unitary conjugations $\rightsquigarrow$ rotations ("U") of the Bloch ball
- depolarizing channel

$$\rho = \frac{1}{2}(I + xX + yY + zZ) \overset{\Delta}{\longmapsto} \frac{I}{2}$$

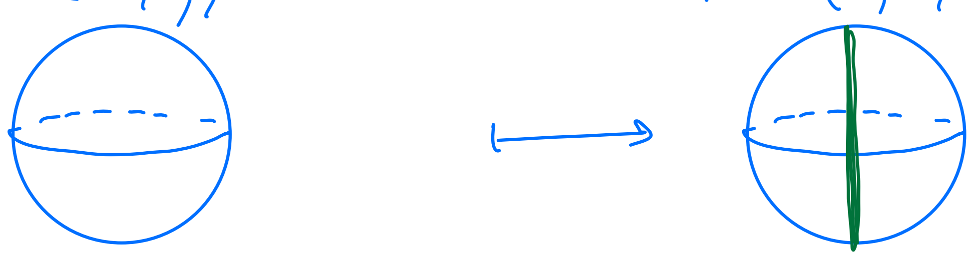
Bloch ball $\vec{r} = (x, y, z)$ $\qquad$ center $\vec{r}' = 0$

- dephasing channel

$$\rho = \frac{1}{2}(I + xX + yY + zZ) \longmapsto \mathrm{diag}(\rho)$$

$$= \frac{1}{2}\begin{bmatrix} 1+z & x+iy \\ x-iy & 1-z \end{bmatrix} \longmapsto \frac{1}{2}\begin{bmatrix} 1+z & 0 \\ 0 & 1-z \end{bmatrix}$$

$$\vec{r} = (x, y, z) \longmapsto \vec{r}' = (0, 0, z)$$



- transposition map

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x+iy \\ x-iy & 1-z \end{bmatrix} \longmapsto \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

$$\vec{r} = (x, y, z) \longmapsto \vec{r}\,' = (x, -y, z)$$

$\leadsto$ it is a reflexion w.r.t. $x-z$ plane

⑤ Structure theorem for quantum channels

→ A q. channel is a TPCP linear map

$$\Phi : M_d \longrightarrow M_D$$

- TP : $\mathrm{Tr}\,\phi(X) = \mathrm{Tr}\,X \quad \forall\, X \in M_d$
- CP : $\forall n$, $\phi \otimes id_n : M_d \otimes M_n \to M_D \otimes M_n$ is positive

(recall : ψ is positive if $X \geq 0 \Rightarrow \psi(X) \geq 0$)

Theorem Let $\Phi : M_d \to M_D$ be a linear map. The following are equivalent:

(1) Φ is a quantum channel

(2) The map $\Phi \otimes id_d$ is TP and positive

(3) The Choi matrix $J(\Phi)$ is PSD and $\mathrm{Tr}_D\, J(\Phi) = I_d$

(4) Kraus decomposition: there exist $A_1, A_2, \dots, A_R \in M_{D\times d}$ such that

$$\phi(X) = \sum_{i=1}^{R} A_i X A_i^* \quad \text{and} \quad \sum_{i=}^{R} A_i^* A_i = I_d$$

(5) Stinespring dilation: there exists an integer R, and an isometry $V : \mathbb{C}^d \longrightarrow \mathbb{C}^D \otimes \mathbb{C}^R$ such that

$$\Phi(X) = [\mathrm{id}_D \otimes \mathrm{Tr}_R](VXV^*)$$

→ The Choi matrix of a linear map $\phi : M_d \to M_D$ is defined as follows:

$$M_D \otimes M_d \ni J(\Phi) = [\Phi \otimes \mathrm{id}_d](d \cdot \omega_d)$$

$$= [\Phi \otimes \mathrm{id}_d]\Big(\sum_{ij=1}^{d} |i\rangle\langle j| \otimes |i\rangle\langle j|\Big)$$

$$= \sum_{ij=1}^{d} \underbrace{\Phi(|i\rangle\langle j|)}_{\in M_D} \otimes \underbrace{|i\rangle\langle j|}_{\in M_d}$$

Examples of Choi matrices

- $\Phi : M_2 \to M_2$ identity channel

$$J(\Phi) = (id_2 \otimes id_2)(2 \cdot \omega_2) = 2 \cdot \omega_2 =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- $\Delta_2 : M_2 \longrightarrow M_2$ depolarizing channel

$$J(\Delta_2) = (\Delta_2 \otimes id_2)\big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \big)$$

$$= \frac{1}{2}\left[I_2 \otimes |0\rangle\langle 0| + I_2 \otimes |1\rangle\langle 1| \right]$$

$$= \frac{I_2}{2} \otimes [\underbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}_{I_2}]$$

$$= \frac{1}{2} I_2 \otimes I_2 = \frac{1}{2} I_4 = \frac{1}{2}\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

→ an isometry $V : \mathbb{C}^k \longrightarrow \mathbb{C}^n$ with the property that $\forall x \in \mathbb{C}^k$ $\|Vx\| = \|x\|$

↳ in particular, if $k = n$ isometry $\equiv$ unitary

↳ equivalently, $V : \mathbb{C}^k \to \mathbb{C}^n$ isometry $\Longleftrightarrow V^* V = I_k$

__Remark__ The integer $R \geq 1$ appearing in (4), (5) is called the Kraus rank of Φ

Moreover, $R = \operatorname{rank} J(\Phi)$

Examples

- identity channel $\quad id : M_d \to M_d$
$$X \mapsto X$$

$\to J(id) = d \cdot \omega \leftarrow$ maximally entangled state

$\to \quad id(X) = X = I \cdot X \cdot I^*$ $\quad$ Kraus decomp.

$R = 1$ and $A_1 = I$

$\to \quad id(X) = X = (id_d \otimes id_1)(I \cdot X \cdot I^*)$

$$V : \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^1 \simeq \mathbb{C}^d$$
$$X \mapsto X \qquad \text{Stinespring decomp}$$

- depolarizing channel $\quad \Delta : M_d \to M_d$
$$X \to (\operatorname{Tr} X) \cdot \frac{I}{d}$$

$\to J(\Delta) = \frac{1}{d} I_{d^2}$ $\quad$ Choi matrix

$$\overset{\shortparallel}{I_d \otimes I_d}$$

$\to \quad \Delta(X) = \frac{1}{d} \sum_{ij} |i\rangle\langle j| \cdot X \cdot (|i\rangle\langle j|)^*$

$\rightsquigarrow R = d^2$ Kraus operators

$A_{ij} = \frac{1}{\sqrt{d}} |i\rangle\langle j|$

$$\frac{1}{d}\sum_{ij} |i\rangle\underbrace{\langle j|\cdot X\cdot|j\rangle}\langle i| = \frac{1}{d}\sum_{ij}\langle j|X|j\rangle\, |i\rangle\langle i|$$

$$= \frac{1}{d}\underbrace{\left[\sum_i |i\rangle\langle i|\right]}_{I_d}\cdot\underbrace{\left[\sum_j \underbrace{\langle j|X|j\rangle}_{X_{jj}}\right]}_{\operatorname{Tr} X}$$

$$= (\operatorname{Tr} X)\,\frac{I_d}{d}$$

- the dephasing channel $\qquad \Phi : M_d \to M_d$
  $\qquad\qquad X \mapsto \operatorname{diag}(X)$

- $\Phi(X) = \operatorname{diag}(X) = \sum_{i=1}^{d} |i\rangle\langle i|\cdot X\cdot(|i\rangle\langle i|)^*$

$\rightsquigarrow$ $R = d$ Kraus operators
$A_i = |i\rangle\langle i|$

$$\sum |i\rangle\langle i|\; X\; \underbrace{(|i\rangle\langle i|)^*}_{|i\rangle\langle i|} = \sum_i |i\rangle\underbrace{\langle i|\cdot X\cdot|i\rangle}\langle i|$$

$$= \sum_i \underbrace{\langle i|X|i\rangle}_{X_{ii}}\, |i\rangle\langle i| = \operatorname{diag}(X)$$

<u>Remark</u> Stinespring decomposition $\rightsquigarrow$ open q. systems.

$\to$ ρ = input = ρ_A. Initially $\rho_{AB} = \rho_A \otimes |0\rangle\langle 0|_B$

→ $(A+B)$ undergoes a unitary evolution

$$\rho'_{AB} = U \rho_{AB} U^* = U(\rho_A \otimes |0\rangle\langle 0|_B) U^*$$

→ Alice's new state is

$$\rho'_A = \mathrm{Tr}_B\, \rho'_{AB}$$

→ From Alice's perspective:

$$\underbrace{\rho_A \longmapsto [\mathrm{id}_A \otimes \mathrm{Tr}_B]\left(U(\rho_A \otimes |0\rangle\langle 0|_B) U^*\right)}_{\Phi(\rho_A) :=}$$

↳ this is the Stinespring dilation of Φ with the isometry

$$V : \mathbb{C}^d \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^R$$

Bob's dim

Alice's dimension

$$x \longmapsto U(x \otimes |0\rangle_B)$$