QUANTUM EVOLUTION

Recall : to a quantum system we associate a Hilbert space (vect. space with an inner product)

$H = \mathbb{C}^d$ d = dimension = # of degrees of freedom

- the states of a q. system = density matrices

$$\{ \rho \in M_{d \times d} : \underbrace{\rho \geq 0}_{} \text{ and } \operatorname{Tr} \rho = 1 \}$$

ρ is PSD (positive semidefinite)
$\Leftrightarrow$ eigenvalues of $\rho \geq 0$

- pure quantum states : rank 1 dens. mat.

$\rho = |\varphi\rangle\langle\varphi|$ with $\varphi \in \mathbb{C}^d$ $\|\varphi\| = 1$

(sometimes we call φ the (pure) state of the q. system)

$\boxed{\text{Axiom 2}}$ The time evolution of a (closed) q. syst is described by a unitary operator.

$\rho' = U \rho U^*$ density matrices

$|\varphi'\rangle = U |\varphi\rangle$ pure states

Physically, the time evolution unitary operator is defined in terms of the Hamiltonian of the system

$$U = \exp\left(-\frac{itH}{\hbar}\right)$$

energy observable

t = time

ρ at $t=0$ $\rightsquigarrow$ ρ' at t

$|\varphi\rangle$ at $t=0$ $\rightsquigarrow$ $|\varphi'\rangle$ at t

Schrödinger equation

$$i\hbar \frac{d|\varphi\rangle}{dt} = H|\varphi\rangle$$

Recall A matrix $U \in M_{d\times d}(\mathbb{C})$ is called unitary if $\forall \vec{v} \in \mathbb{C}^d$, $\|U\vec{v}\| = \|\vec{v}\|$.

Equivalent characterizations

- U is unitary $\|U\vec{v}\| = \|\vec{v}\| \quad \forall \vec{v} \in \mathbb{C}^d$
- $UU^* = U^*U = I_d$
- $\langle U\vec{v}, U\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle \quad \forall \vec{v}, \vec{w} \in \mathbb{C}^d$

$$\langle \vec{v}, U^*U\vec{w}\rangle = \langle \vec{v}, I\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle$$

since $\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A^*\vec{y}\rangle$

recall : $(A^*)_{ij} = (\bar{A})_{ji}$

$$A^* = (\bar{A})^T$$

Examples

- identity matrix $I_d = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

- a diagonal matrix $A = \mathrm{diag}(a_1, a_2, \dots, a_d)$ is unitary if and only if $|a_i| = 1 \ \forall i$

$$A^* A = \mathrm{diag}(\bar{a}_i) \cdot \mathrm{diag}(a_i) = \mathrm{diag}(|a_i|^2)$$

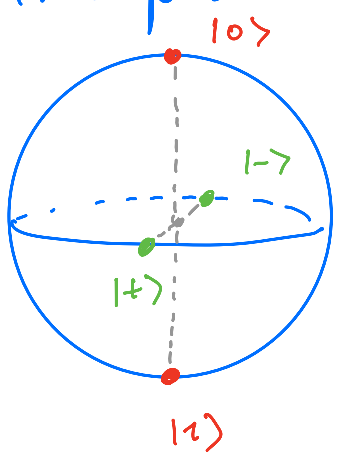
- a non-example: O_d ; $O^* O = O \neq I$

- Hadamard matrix (Hadamard gate)

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \in M_2(\mathbb{C})$$

$H^* = H$ so $H^* H = H^2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I$

Let us denote by $\{|0\rangle, |1\rangle\}$ the canonical basis of the qubit Hilbert space.



Bloch ball

$H|0\rangle = 1^{st}$ column of $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = |+\rangle$

$H|1\rangle = 2^{nd}$ column of $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) = |-\rangle$

- Pauli matrices are unitary!

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$X|0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$X|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

So X is the quantum analogue of the NOT operation from logic

$$Y|0\rangle = \begin{bmatrix} 0 \\ i \end{bmatrix} = i|1\rangle$$

$$Y|1\rangle = \begin{bmatrix} -i \\ 0 \end{bmatrix} = -i|0\rangle$$

$$Z|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$Z|1\rangle = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -|1\rangle$$

So Z introduces a phase of -1 in front of $|1\rangle$

Let us check that these matrices are unitary:

- $X^* = X$, $Y^* = Y$, $Z^* = Z$

the Pauli matrices are self-adjoint

$$X^*X = X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

"NOT ∘ NOT = identity"

Eigenvalues of unitary operators

$U^* U = U U^* = I \Rightarrow U$ is normal

$\Rightarrow$ we have a spectral decomposition

$$U = V \cdot D \cdot V^*$$

where V is unitary

D is diagonal, containing the eigenvalues

Then $U^* = V \cdot \underset{\overline{D}}{\underset{\parallel}{D^*}} \cdot V^*$ since D is diagonal

$$U^* U = V \overline{D} \underbrace{V^* \cdot V}_{I} D V^* = V \cdot D' \cdot V^*$$

where $D' = \operatorname{diag}(|\lambda_i|^2)$

if $D = \operatorname{diag}(\lambda_i)$

$\uparrow$ eigenvalues of U

So: $I = V D' V^*$

multiply with V^* from the left and with V from the right

$$\underbrace{V^* \cdot I \cdot V}_{I} = \underbrace{V^* \cdot V}_{I} D' \underbrace{V^* \cdot V}_{I} \Rightarrow D' = I$$

$\Rightarrow |\lambda_i| = 1$ for $i = 1, 2, \dots, d$

$\boxed{\text{The eigenvalues of unitary operators are phases (complex numbers of modulus 1)}}$

Examples

- H is unitary, self-adjoint
  $\Rightarrow$ eigenvalues are either $+1$ or -1
  $\mathrm{Tr}\, H = 0 \Rightarrow$ eigenvalues are $+1$ and -1

- the same applies to Pauli matrices:

$$X|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) =$$

$$= \frac{1}{\sqrt{2}} \left(|1\rangle + |0\rangle \right) = |+\rangle$$

eigenvalue 1 ; eigenvector $|+\rangle$

$$X|-\rangle = -|-\rangle$$

eigenvalue -1 ; eigenvector $|-\rangle$

QUANTUM MEASUREMENT

Postulates of Q. Mechanics

- Hilbert space: each q. syst comes with a Hilbert space
  $H = \mathbb{C}^d$

- States of a q. syst: density matrices
  $$\{ \rho \in \mathcal{M}_d(\mathbb{C}) : \rho \geq 0 \quad \mathrm{Tr}\, \rho = 1 \}$$
  ↳ pure states $\quad \rho = |\varphi\rangle\langle\varphi| \quad$ for $\varphi \in \mathbb{C}^d$ $\|\varphi\| = 1$

- time evolution: unitary matrices $U^*U = U^*U = I$
  $$\rho' = U \rho U^* \quad ; \quad |\varphi'\rangle = U|\varphi\rangle$$

- quantum measurement

Definition A measurement device is associated to a self-adjoint operator acting on H called a quantum observable.

$$A \in \mathcal{M}_d(\mathbb{C}) \qquad A = A^*$$ ⤳ A is normal

- spectral decomposition
  $$A = \sum_{i=1}^{d} \lambda_i P_i$$

eigenvalues $\qquad$ eigenprojections

- when one measures a system in state ρ, one obtains one of the results $\{\lambda_1, \lambda_2, \dots, \lambda_d\}$ with probability

$$\mathbb{P}[\text{observe } \lambda_i] = \mathrm{Tr}[\rho \cdot P_i]$$

⚠ The result of a quantum measurement is random

<u>Example</u> : measuring in the canonical basis

- $\mathcal{H} = \mathbb{C}^2$ a qubit
- $A = 1 \cdot |0\rangle\langle 0| + 2 \cdot |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\rightsquigarrow$ 2 outcomes $\{1, 2\}$

The labels of the outcomes (the eigs. of A) do not really matter.

$B = 5 \cdot |0\rangle\langle 0| + (-\pi) \cdot |1\rangle\langle 1|$

$\rightsquigarrow$ A and B are basically the same q. observable, up to a relabelling of the classical outcomes.

What is important: eigenprojections.

- If we measure a q. system in a state $\rho \in \mathcal{M}_2(\mathbb{C})$

$$\mathbb{P}[\text{observe "1"}] = \mathrm{Tr}[\rho \cdot |0\rangle\langle 0|]$$
$$= \langle 0|\rho|0\rangle$$
$$\mathbb{P}[\text{observe "2"}] = \mathrm{Tr}[\rho \cdot |1\rangle\langle 1|]$$
$$= \langle 1|\rho|1\rangle.$$

- if $\rho = \frac{1}{2}[I + xX + yY + zZ]$

Bloch ball representation

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x+iy \\ x-iy & 1-z \end{bmatrix}$$

$$\mathbb{P}[\text{"1"}] = \langle 0|\rho|0\rangle = \frac{1}{2}(1+z)$$
$$\mathbb{P}[\text{"2"}] = \langle 1|\rho|1\rangle = \frac{1}{2}(1-z)$$

note that $\frac{1}{2}(1+z), \frac{1}{2}(1-z) \geq 0$
and $\frac{1}{2}(1+z) + \frac{1}{2}(1-z) = 1$
(it is a probability mass function)

- importantly, we have measured A, but we could have also measured

$$Z = Z^* = 1 \cdot |0\rangle\langle 0| + (-1) \cdot |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- if we measure the X Pauli matrix

$$X = X^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 1 \cdot \underbrace{|+\rangle}_{= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)} + (-1) \underbrace{|-\rangle}_{= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}$$

↳ if we measure X on ρ

$$\mathbb{P}[\text{"}+1\text{"}] = \mathrm{Tr}[\rho \cdot |+\rangle\langle+|] = \langle+|\rho|+\rangle$$

$$= \frac{1}{2}\left[\langle 0|\rho|0\rangle + \langle 0|\rho|1\rangle + \langle 1|\rho|0\rangle + \langle 1|\rho|1\rangle\right]$$

$$= \frac{1}{2}(1+x)$$

$$\mathbb{P}[\text{"}-1\text{"}] = \frac{1}{2}(1-x)$$

↳ similarly for Y

$$\mathbb{P}(\text{"}\pm 1\text{"}) = \frac{1}{2}(1 \pm y)$$

- if we measure Z on $\rho = |+\rangle\langle+|$

$$\mathbb{P}[\text{"}1\text{"}] = \langle 0|\rho|0\rangle = \langle 0| \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} |0\rangle$$

$$= \frac{1}{2}$$

$$\mathbb{P}[\text{"}-1\text{"}] = \langle 1|\rho|1\rangle = \frac{1}{2}$$

When measuring the q. observable Z on a q. syst in state $\rho = |+\rangle\langle+|$, we get the outcomes ± 1 with equal probabilities $\frac{1}{2}$

- if we measure Z on $\rho = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\mathbb{P}[\text{"1"}] = \langle 0|\rho|0\rangle = 1$$

$$\mathbb{P}[\text{"}-1\text{"}] = \langle 1|\rho|1\rangle = 0$$

In this case, the result of the q. measurement in NOT random

- Similarly, if $\rho = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathbb{P}[\text{"1"}] = \langle 0|\rho|0\rangle = 0$$

$$\mathbb{P}[\text{"}-1\text{"}] = \langle 1|\rho|1\rangle = 1$$

Here, we always get outcome "-1"

Fact

If we measure an observable A on a quantum system is state $\rho = P_k$ of rank 1 (P_k is an eigenprojector of A), the outcome will be k with prob. 1.

Proof

$$A = \sum_i \lambda_i P_i \rightsquigarrow P_i \text{ are orthogonal}$$

$$Tr(P_i P_j) = 0 \text{ if } i \neq j$$

$$\mathbb{P}[\text{outcome } i] = Tr[\overset{P_k}{\overset{\shortparallel}{\rho}} \cdot P_i]$$

$$= \begin{cases} Tr[P_k^2] = 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$\rightsquigarrow$ we always obtain outcome k