Completely Positive Matrices and Quantum Entanglement

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Talk outline

Diagonal unitary covariant maps

Completely positive matrices, generalizations, and quantum entanglement

Factor width

The PPT^2 conjecture

Diagonal unitary covariant maps

Main definition

- Let $\mathcal{U}_d$ be the set of $d \times d$ unitary operators. A bipartite matrix $X \in \mathcal{M}_d \otimes \mathcal{M}_d$ is invariant by the conjugate action of any $U \in \mathcal{U}_d$, i.e. $\forall U \in \mathcal{DU}_d$, $(U \otimes \bar{U})X(U \otimes \bar{U})^* = X$ iff X belongs to the span of the identity matrix and the maximally entangled state $\omega = \sum_{i,j=1}^{d} |ii\rangle\langle jj|$
- Let $\mathcal{DU}_d$ be the set of diagonal unitary operators

$$U = \mathrm{diag}(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_d}), \qquad \theta_j \in \mathbb{R}$$

and $\mathcal{DO}_d$ be the set of diagonal orthogonal operators

$$U = \mathrm{diag}(\pm 1, \ldots, \pm 1)$$

Definition

A bipartite matrix $X \in \mathcal{M}_d \otimes \mathcal{M}_d$ is said to be

- conjugate diagonal unitary covariant (CDUC) if $\forall U \in \mathcal{DU}_d$

$$(U \otimes \bar{U})X(U \otimes \bar{U})^* = X$$

- diagonal orthogonal covariant (DOC) if $\forall U \in \mathcal{DO}_d$

$$(U \otimes U)X(U \otimes U)^\top = X$$

Explicit form

Proposition ([SN21])

The vector space of CDUC matrices is parametrized by pairs of matrices $(A, B) \in \mathcal{M}_d^2$ having the same diagonal $\operatorname{diag} A = \operatorname{diag} B$:

$$X_{A,B}(ij, kl) = \delta_{ik}\delta_{jl}A_{ij} + \delta_{ij}\delta_{kl}B_{ij} - \delta_{ijkl}A_{ii}$$

The vector space of DOC matrices is parametrized by triples of matrices $(A, B, C) \in \mathcal{M}_d^3$ having the same diagonal $\operatorname{diag} A = \operatorname{diag} B = \operatorname{diag} C$:

$$X_{A,B,C}(ij, kl) = \delta_{ik}\delta_{jl}A_{ij} + \delta_{ij}\delta_{kl}B_{ij} + \delta il jk C_{ij} - 2\delta_{ijkl}A_{ii}$$

- The identity matrix is CDUC: $A = J$, $B = I$
- The maximally entangled state is CDUC: $A = I$, $B = J$
- The flip operator $F(x \otimes y) = y \otimes x$ is DOC: $A = B = I$, $C = J$
- The Choi matrix of the Choi map $\Phi_{\mathrm{Choi}} : \mathcal{M}_3 \to \mathcal{M}_3$ is CDUC

$$\Phi_{\mathrm{Choi}}(X) = \begin{bmatrix} X_{11} + X_{33} & -X_{12} & -X_{13} \\ -X_{21} & X_{11} + X_{22} & -X_{23} \\ -X_{31} & -X_{32} & X_{22} + X_{33} \end{bmatrix}$$

Properties of CDUC matrices

- The positivity properties of CDUC matrices depend on the convex cones

$$\mathrm{EWP}_d = \{A \in \mathcal{M}_d \ : \ A_{ij} \geq 0 \quad \forall i,j\}$$

$$\mathrm{PSD}_d = \{B \in \mathcal{M}_d \ : \ B \text{ is positive semidefinite, i.e. } B = ZZ^*\}$$

- A bipartite matrix $X \in \mathcal{M}_d \otimes \mathcal{M}_d$ is said to have positive partial transpose (PPT) if $X \in \mathrm{PSD}_{d^2}$, and, moreover

$$X^\Gamma := [\mathrm{id} \otimes \mathrm{transp}](X) \in \mathrm{PSD}_{d^2}$$

- Example: the maximally entangled state $\omega = \sum_{ij} |ii\rangle\langle jj|$ is not PPT

Proposition ([SN21])

Let $A, B \in \mathcal{M}_d$ with $\mathrm{diag}\, A = \mathrm{diag}\, B$. Then

- $X_{A,B}$ *is PSD* $\iff$ $A \in \mathrm{EWP}_d$ *and* $B \in \mathrm{PSD}_d$
- $X_{A,B}$ *is PPT* $\iff$ $A \in \mathrm{EWP}_d$, $B \in \mathrm{PSD}_d$ *and* $A_{ij}A_{ji} \geq |B_{ij}|^2 \quad \forall i,j$
- $[\mathrm{id} \otimes \mathrm{Tr}]X_{A,B} = I_d \iff \sum_j A_{ij} = 1 \quad \forall i$
- $[\mathrm{Tr} \otimes \mathrm{id}]X_{A,B} = I_d \iff \sum_i A_{ij} = 1 \quad \forall j$

Completely positive matrices, generalizations, and quantum entanglement

Completely positive matrices

Definition ([BSM03])

A matrix $A \in \mathcal{M}_d$ is said to be completely positive (CP) if it can be written as

$$A = ZZ^\top, \qquad \text{with } Z \in \mathrm{EWP}_d$$

In other words, A is CP if there exist finitely many vectors $v_n \in \mathbb{C}^d$ s.t.

$$A = \sum_n |v_n \odot \overline{v_n}\rangle\langle v_n \odot \overline{v_n}|$$

- The cone of completely positive matrices (and its dual, the cone of completely copositive matrices) has many uses in applied mathematics and optimization
- Clearly, $\mathrm{CP}_d \subseteq \mathrm{PSD}_d \cap \mathrm{EWP}_d$, the inclusion being strict for $d \geq 5$

Separable matrices

Definition ([HHHH09])

A bipartite matrix $X \in \mathcal{M}_d \otimes \mathcal{M}_d$ is said to be separable (SEP) if it can be written as

$$X = \sum_n Y_n \otimes Z_n \quad \text{with } Y_n, Z_n \in \mathsf{PSD}_d$$

In other words, the cone of separable matrices is spanned by tensor products of positive semidefinite matrices.
Non-separable matrices are called entangled

- We have the following chain of inclusions:

$$\mathrm{SEP}_d \subseteq \mathrm{PPT}_d \subseteq \mathsf{PSD}_{d^2}$$

- The first inclusion is strict for $d \geq 3$ [HHH96]
- Deciding membership in SEP_d is NP-hard

Separable CDUC matrices and the PCP cone

Definition (Johnston and MacLean [JM19])

A pair $(A, B) \in \mathcal{M}_d^2$ is said to be pairwise completely positive (PCP) if there exist finitely many vectors $v_n, w_n \in \mathbb{C}^d$ such that

$$A = \sum_n |v_n \odot \overline{v_n}\rangle\langle w_n \odot \overline{w_n}| \qquad B = \sum_n |v_n \odot w_n\rangle\langle v_n \odot w_n|$$

Proposition ([SN21])

Let $A, B \in \mathcal{M}_d$ *with* $\operatorname{diag} A = \operatorname{diag} B$. *Then*

$$X_{A,B} \in \mathrm{SEP}_d \iff (A, B) \in \mathrm{PCP}_d$$

- A pair (A, A) is PCP iff A is completely positive. In particular,

$$X_{A,A} \in \mathrm{SEP}_d \iff A \in \mathrm{CP}_d$$

- Similar deifinition and result for DOC matrices $\rightsquigarrow$ TCP_d cone
- Deciding membership in CP_d, PCP_d, and TCP_d is NP-hard

Factor width

Factor width

Definition ([BCPT05])

A matrix $B \in \mathrm{PSD}_d$ is said to have factor width k if it admits a rank one decomposition $B = \sum_n |z_n\rangle\langle z_n|$, such that, for all n, $\#\,\mathrm{supp}(z_n) \le k$

Definition ([SN20])

A matrix pair $(A, B) \in \mathrm{PCP}_d$ is said to have factor width k if it admits a *PCP* decomposition $A = \sum_n |v_n \odot \overline{v_n}\rangle\langle w_n \odot \overline{w_n}|$ and $B = \sum_n |v_n \odot w_n\rangle\langle v_n \odot w_n|$ with $\#\,\mathrm{supp}(v_n \odot w_n) \le k$ for all n

- The sets above are denoted by PSD_d^k, resp. PCP_d^k
- We have the following inclusions

$$\mathrm{PSD}_d^1 \subseteq \mathrm{PSD}_d^2 \subseteq \cdots \subseteq \mathrm{PSD}_d^d = \mathrm{PSD}_d$$
$$\mathrm{PCP}_d^1 \subseteq \mathrm{PCP}_d^2 \subseteq \cdots \subseteq \mathrm{PCP}_d^d = \mathrm{PCP}_d$$

- PSD_d^1 is the set of diagonal matrices in EWP_d
- PCP_d^1 is the set of matrix pairs $(A, B) \in \mathrm{PCP}_d$ such that $B = \mathrm{diag}\, A$

Factor width two

- To any matrix B, associate its comparison matrix

$$M(B)_{ij} = \begin{cases} |B_{ij}| & \text{if } i = j \\ -|B_{ij}| & \text{otherwise} \end{cases}$$

Proposition ([BCPT05])

For a (hermitian) matrix $B \in \mathcal{M}_d$, the following equivalences hold:

$$B \in \mathrm{PSD}_d^2 \iff M(B) \in \mathrm{PSD}_d \iff B \text{ is scaled diagonally dominant}$$

Proposition ([SN20])

For $A, B \in \mathcal{M}_d$ such that

$$A \in \mathrm{EWP}_d \quad \text{and} \quad B \in \mathrm{PSD}_d \quad \text{and} \quad A_{ij}A_{ji} \geq |B_{ij}|^2 \quad \forall i, j$$

the following equivalence holds:

$$(A, B) \in \mathrm{PCP}_d^2 \iff B \in \mathrm{PSD}_d^2 \iff M(B) \in \mathrm{PSD}_d$$

$\rightsquigarrow$ A simple criterion for membership inside $\mathrm{PCP}_d^2 \subseteq \mathrm{PCP}_d (\leftrightarrow \text{SEP})$

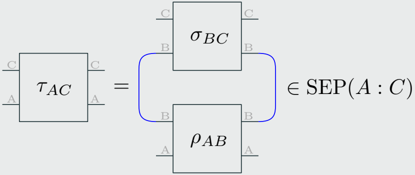
The PPT^2 conjecture

Statement of the conjecture

- Informally, the PPT^2 conjecture states that any pair of PPT states, when combined in a certain way, yield a separable state
- The precise way of combining the matrices corresponds to the composition of the corresponding quantum (PPT) channels, to yield an entanglement breaking channel
- This conjecture is relevant for quantum information because it imposes constraints on the type of resources that can be distributed using quantum repeaters, a key element of quantum internet [BCHW15, CF17]

Conjecture ([Chr12])

Given a pair of bipartite matrices $\rho_{AB} \in \mathrm{PPT}(A:B)$ *and* $\sigma_{BC} \in \mathrm{PPT}(B:C)$, *it holds that*



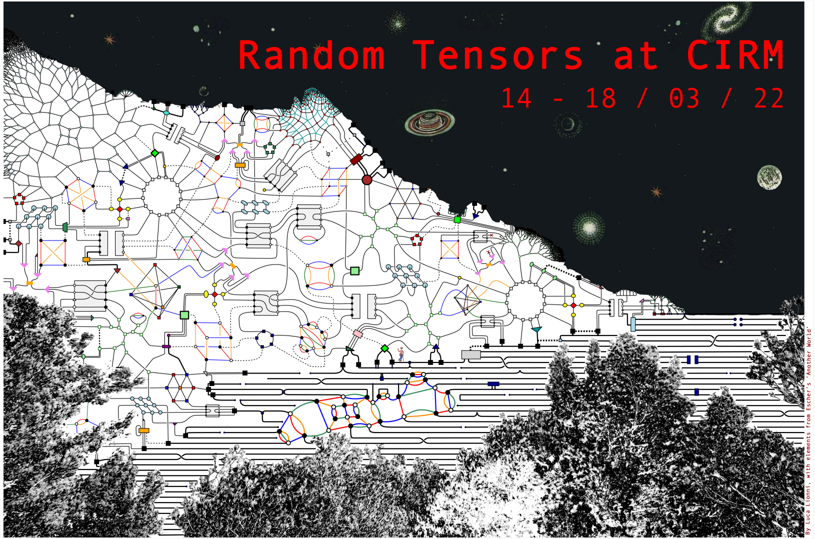
Progress on the proof

- Trivially holds for qubits since $\mathrm{PPT} \iff \mathrm{SEP}$ for $d = 2$
- The distance between iterates of a unital (or trace preserving) PPT map and the set of entanglement breaking maps tends to zero in the asymptotic limit [KMP18]
- Any unital (or trace preserving) PPT map becomes entanglement breaking after finitely many iterations of composition with itself [RJP18]
- For other algebraic approaches and extensions, see [LG15, HRF20, GKS20]
- The conjecture holds for fully unitary covariant channels [VW01, CMHW19]
- Independent random quantum channels satisfy the conjecture [CYZ18]
- Gaussian maps satisfy the conjecture [CMHW19]
- The conjecture holds for qutrits [CYT19, CMHW19]

Theorem ([SN20])

PPT^2 holds for CDUC matrices

⇝ full proof claimed recently by A. Majewski arXiv:2108.01588



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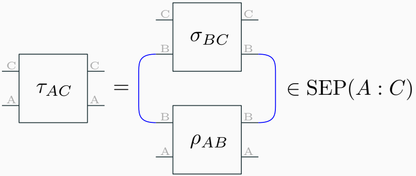
The take-home slide

- Bipartite matrices X covariant under the action of the diagonal unitary group: $\forall U \in \mathcal{DU}_d$, $(U \otimes \bar{U})X(U \otimes \bar{U})^* = X$ are called CDUC
- $X_{A,B}(ij,kl) = \delta_{ik}\delta_{jl}A_{ij} + \delta_{ij}\delta_{kl}B_{ij} - \delta_{ijkl}A_{ii}$
- $X_{A,B}$ is PSD $\iff$ $A \in \mathrm{EWP}_d$ and $B \in \mathrm{PSD}_d$. It is PPT if, moreover, $A_{ij}A_{ji} \geq |B_{ij}|^2 \quad \forall i,j$. It is SEP $\iff$ $(A,B) \in \mathrm{PCP}_d$:

$$A = \sum_n |v_n \odot \overline{v_n}\rangle\langle w_n \odot \overline{w_n}| \qquad B = \sum_n |v_n \odot w_n\rangle\langle v_n \odot w_n|$$

- The PPT^2 conjecture: given a pair of bipartite matrices $\rho_{AB} \in \mathrm{PPT}(A:B)$ and $\sigma_{BC} \in \mathrm{PPT}(B:C)$, it holds that



- Main result: PPT^2 holds for CDUC matrices
- No simple criterion for memb. in TCP_d: PPT^2 open for DOC matreices

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