

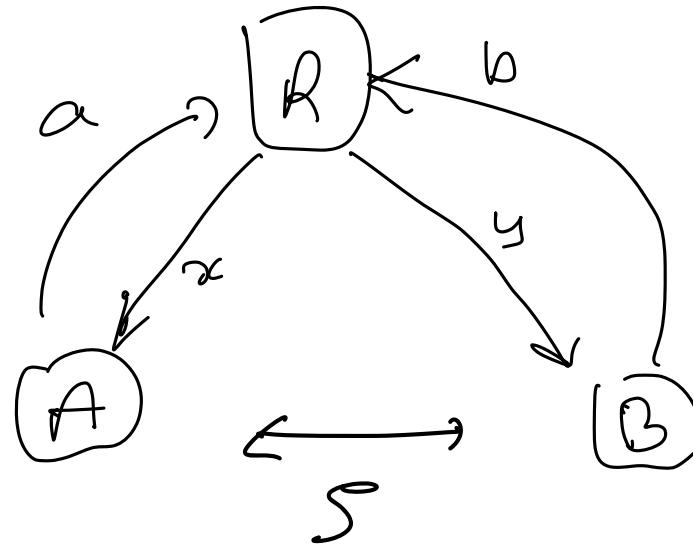
Tensor norms & applications to QIT

(joint with Faedri Loulidi)

①

Motivation

2 player XOR game, n q / 2 a



Players want to maximize

$$P_M(\rho, A, B) = \sum_{x, y \in [n]} M_{xy} \text{Tr}(\rho \cdot A_x \otimes B_y)$$

$$A_x = E_{+1x} - E_{-1x}$$

$$B_y = F_{+1y} - F_{-1y}$$

$$\{E_{\pm 1x}\}_{x \in [n]} \quad \{F_{\pm 1y}\}_{y \in [n]}$$

Example $n=2 \quad M = M_{CHSH} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Theorem [Wolf, Perez-Garcia, Fernandez '19]

Fix A_1, A_2 observables $\in \mathbb{M}_d$.

$$\beta_M(A_1, A_2) := \sup_{\rho} \sup_{\text{state}} \sup_{\mathcal{I} \subseteq B_{1,2} \subseteq I} \beta_M(\rho, A, B)$$

$P_{\text{MCHSH}}(A) > 1 \Leftrightarrow A_1, A_2$ incompatible

Viol of CHSH \Leftarrow incompatibility

Goal • explain this / c-ex

- other Bell ineq
- relate amount of viol to incorp. robustness

②

Tensor norms

$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ f.d. real Banach spaces

~ which norm on $X \otimes Y$?

(H1) $\|x \otimes y\| = \|x\|_X \cdot \|y\|_Y$

$$\rightarrow X^* = \{ \varphi : X \rightarrow \mathbb{R} \}$$

$$\|\varphi\|_{X^*} = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\varphi(x)|$$

(H2) $\|\varphi \otimes \psi\|_* = \|\varphi\|_{X^*} \cdot \|\psi\|_{Y^*}$

A tensor norm is a norm $\|\cdot\|$ on $X \otimes Y$ satisfying (H1) and (H2)

Example $\ell_2^m = (\mathbb{R}^m, \|\cdot\|_2)$

- $\|\cdot\|_2$ is a tensor norm on $\ell_2^m \otimes \ell_2^n$
but $\mathbb{R}^m \otimes \mathbb{R}^n \cong M_{m \times n}(\mathbb{R})$
- $\|\cdot\|_{op} = \|\cdot\|_2$ is a tensor norm on $\ell_2^m \otimes \ell_2^n \cong M_{m \times n}(\mathbb{R})$
 $\|x \otimes y\|_{op} = \|x\|_2 \cdot \|y\|_2$
- its dual $\|\cdot\|_7 = \|\cdot\|_{tr.}$ is also a tensor norm

Theorem $\| \cdot \|_{\text{tensor norm}} \|$ on $X \otimes Y$

$$\| \cdot \|_{\Sigma} \leq \| \cdot \| \leq \| \cdot \|_{\pi}$$

where $\| z \|_{\Sigma} = \sup_{\|\varphi\|_{X^*} \leq 1} |\langle \varphi \otimes \psi, z \rangle|$

$$\|\varphi\|_{X^*} \leq 1$$

$$\|\psi\|_{Y^*} \leq 1$$

$$\| z \|_{\pi} = \inf \left\{ \sum_{i=1}^k \|x_i\| \|y_i\| : \right.$$

$$z = \sum_{i=1}^k x_i \otimes y_i \left. \right\}$$

Rk: $\| \cdot \|_{op} = \| \cdot \|_{\Sigma} ; \| \cdot \|_{tr} = \| \cdot \|_{\pi}$

Rk

$$X \in \text{Re}_d^{sa}(\mathbb{C})$$

$$X \geq 0 \Leftrightarrow \|X\|_1 = \text{Tr} X$$

$$X \in \mathcal{M}_{d_A}^{\text{sa}} \otimes \mathcal{M}_{d_B}^{\text{sa}}$$

$$X \in \text{SEP} \Leftrightarrow \|X\|_{S_1^{\frac{d_A}{2}} \otimes S_1^{\frac{d_B}{2}}} = \text{Tr } X$$

entanglement via tensor norms

arXiv: 2010.06365

③ XOR games

Alice uses n POUFs (2 outcomes)

$$(E_{\pm 1})_1, \dots, (E_{\pm 1^n})$$

$$\hookrightarrow A := \sum_{x=1}^n |x\rangle \otimes A_x \in \mathbb{R}^{\otimes n} \otimes \mathcal{M}_d^{\text{sa}}$$

$$A_x = E_{+x} - E_{-x}$$

$\rightarrow A \in \mathbb{R}^n \otimes M_d^{S\otimes}$ corresponds to dos.

$$\Leftrightarrow \|A\|_{\ell_\infty^n \otimes S_d^\otimes} \leq 1$$

fact $\left\| \sum_i |i\rangle \otimes |x_i\rangle \right\|_{\ell_\infty \otimes \sum} = \max_i \|x_i\|_X$

$$\left\| \sum_i |x_i\rangle \otimes |i\rangle \right\|_{\ell_1 \otimes \pi} = \sum_i \|x_i\|_X$$

Theorem 1 (Loulidi - N 2021)

Let $M \in M_n(\mathbb{R})$, invertible. Then

$$\|A\|_M := \sup_P \sup_{\|B\|_1 \leq 1} \sum_{x,y=1}^n \text{Tr}_{xy} \langle P, A_x \otimes B_y \rangle$$

$$= \lambda_{\max} \sum_y \left| \sum_x M_{xy} A_x \right|$$

is a tensor norm on

$$(R^m, \| \cdot \|_M) \otimes S_\infty^d$$

$$\| a \|_M = \sum_y \left(\sum_x M_{xy} a_x \right)$$

→ max classical bias of M :

$$\beta(M) = \sup_{a_x = \pm 1} \| a \|_M =$$

$$= \sup_{\substack{a_x = \pm 1 \\ b_y = \pm 1}} \sum_{xy} M_{xy} a_x b_y$$

$$= \sup_{\substack{\text{all } a \leq 1 \\ \|b\|_\infty \leq 1}} \langle m, a \otimes b \rangle$$

$$\geq \|m\|_{\ell_1^n \otimes \ell_1^n}$$

\hookrightarrow assume $\|m\|_{\ell_1 \otimes \ell_1} = 1$ from now on

Cor Alice has fixed obs A_1, \dots, A_n . Then, we can see viol of $m \Leftrightarrow \|A\|_M \geq 1$

④ Compatibility of q. meas

Theorem 2 [Dlukhm, Jencova, N '2021]

$(E^{\pm|x})_{x \in \{a\}}$ comp (\Rightarrow)

$\|A\|_{\text{comp}} \leq 1$ where

$$\|A\|_{\text{comp}} = \inf \left\{ \lambda_{\max} \sum_{i=1}^k h_i : \right.$$

$$A = \sum z_i \otimes h_i, \quad \left. \begin{aligned} & \|z_i\|_\infty \leq 1 \text{ and } \\ & h_i \geq 0 \end{aligned} \right\}$$

Example

Pauli measurements

$$E_{\pm i} = \frac{1}{2} (I \pm x_i \sigma_i)$$

$$A_i = x_i \sigma_i$$

$$\left\| \sum_{i=1}^3 |i\rangle \otimes x_i \sigma_i \right\|_{\text{comp}} = \left\| (x_1, x_2, x_3) \right\|_2$$

⑤

Main result

Theorem 3 (Coulidi - N 2021)

Let M invertible s.t. $\|M\|_{\ell_1 \otimes \ell_1} = 1$

Then , & A_1, \dots, A_n obs. ,

$$\|A\|_M \leq \|A\|_S$$

non-locality \Rightarrow incompatibility

Moreover, if

$$\|N^{-1}\|_{\ell_\infty \otimes \ell_\infty} = \max_{i,j \in [n]} |(N^{-1})_{ij}| \leq 1$$

then $\|A\|_S \leq \|A\|_M$

non-locality (\Rightarrow) incompatibility

Example • $M_{CHSH} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is such that

$$\|\cdot\|_S = \|\cdot\|_{CHSH}$$

• M_T is not like that :

$I_{3322, \text{corr}}$

$$\exists A \text{ s.t. } \|A\|_{I_{3322, \text{corr}}} < \|A\|_p$$

i.e. incomp. observables which do
not violate $I_{3322, \text{corr}}$

$$M_{3322, \text{corr}} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$