Measurement incompatibility vs. Bell non-locality

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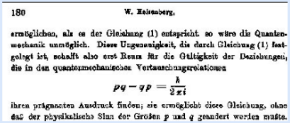
NanoX-FeRMI days, March 10th 2022

Outline

Quantum incompatibility

There exist quantum observables which cannot be measured simultaneously


180 W. Heisenberg,

ermöglichen, als es der Gleichung (1) entspricht, so wäre die Quantenmechanik unmöglich. Diese Ungenauigkeit, die durch Gleichung (1) festgelegt ist, schafft also erst Raum für die Gültigkeit der Beziehungen, die in den quantenmechanischen Vertauschungsrelationen

$$pq - qp = \frac{h}{2\pi i}$$

ihren prägnanten Ausdruck finden; sie ermöglicht diese Gleichung, ohne daß der physikalische Sinn der Größen p und q geändert werden müßte.

always

for certain Bell inequalities

Bell non-locality

The quantum value of certain correlation inequalities is strictly larger than the classical value

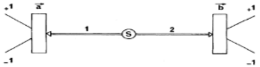


FIG. 1. Einstein–Podolsky–Rosen–Bohm *gedankenexperiment*. Two-spin-$\frac{1}{2}$ particles (or photons) in a singlet state (or similar) separate. The spin components (or linear polarizations) of 1 and 2 are measured along $\vec{a}$ and $\vec{b}$. Quantum mechanics predicts strong correlations between these measurements.

We relate, in a quantitative manner, the incompatibility of N quantum measurements to the largest violation of a given Bell inequality, when one party uses the N measurements.

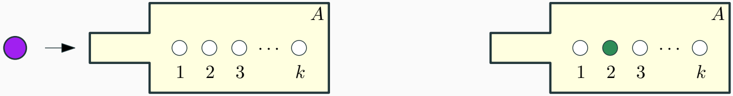
Measurement compatibility

Quantum measurements

- In quantum mechanics, the measurement postulate describes the outcome probabilities and the posterior state when measuring an observable X on a quantum state described by a d dimensional density operator ρ.
- Outcome probabilities are given by the Born rule: $\mathbb{P}[\,\text{outcome } i\,] = \operatorname{Tr} \rho P_i$, where P_i are the eigenprojections of the observable X.
- Allowing for more general measurement scenarios (interaction with an ancilla system), one can describe measurement outcomes in the POVM (Positive-Operator-Valued Measure) framework [NC10, Wat18]:

$$\mathbb{P}[\,\text{outcome } i\,] = \operatorname{Tr} \rho A_i,$$

where $A_1, \ldots, A_k$ are positive semi-definite matrices s.t. $\sum_{i=1}^k A_i = I_d$.



A particle enters a measurement apparatus described by a POVM $A = (A_1, \ldots, A_k)$. The measurement yields outcome $i = 2$, with prob. $\operatorname{Tr} \rho A_2$.

Measurement (in-)compatibility

- In quantum mechanics, there exist incompatible quantum measurements, i.e. measurements that cannot be simultaneously performed on a single copy of the system.
- In the general framework of POVMs, two measurements A, B are said to be compatible [HMZ16] if there exists a third measurement $C = (C_{ij})$ such that A and B are the marginals of C:

$$\forall i = 1, \ldots, k, \qquad A_i = \sum_{j=1}^{l} C_{ij}$$

$$\forall j = 1, \ldots, l, \qquad B_j = \sum_{i=1}^{k} C_{ij}$$

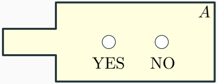
- Projective measurements are compatible if and only if they commute:

$$C_{ij} = A_i B_j = \sqrt{A_i} B_j \sqrt{A_i} = \sqrt{B_j} A_i \sqrt{B_j}$$

- In many scenarios in quantum information theory, measurement incompatibility is a necessary ingredient to obtain non-classical effects.

Norm characterization

Dichotomic POVMs (A_1, A_2) correspond to measurements with two possible outcomes: YES / NO measurements. To such a POVM, we associate a measurement operator $(A_1, A_2) \mapsto A := A_1 - A_2$ with $-I_d \leq A \leq I_d$.



Theorem ([BJN20])

A tuple $A = (A_1, A_2, \dots, A_N) = \sum_{i=1}^{N} |i\rangle \otimes A_i$ of measurement operators corresponds to compatible POVMs $\iff \|A\|_c \leq 1$, with

$$\|A\|_c := \inf \left\{ \left\| \sum_{j=1}^{K} H_j \right\|_\infty : A = \sum_{j=1}^{K} z_j \otimes H_j, \ \text{s.t.} \ \|z_j\|_\infty \leq 1 \text{ and } H_j \geq 0 \right\}$$

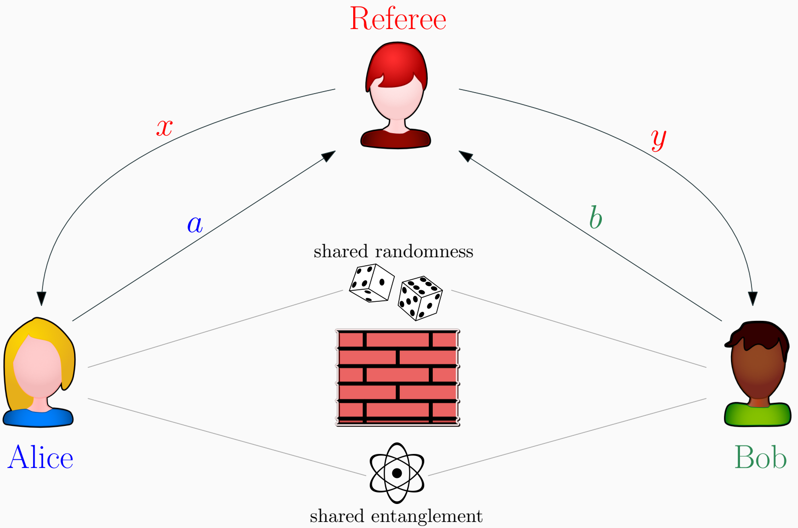
where $z_j \in \mathbb{R}^N$ and $H_j \in \mathcal{M}_d^{sa}(\mathbb{C})$; $\|\cdot\|_c$ is called the compatibility norm.

- As an example, consider noisy measurements in the Pauli bases: $A_X = x\sigma_X$, $A_Y = y\sigma_Y$, $A_Z = z\sigma_Z$. The compatibility norm reads in this case $\|(A_X, A_Y, A_Z)\|_c = \|(x, y, z)\|_2$ [Bus86].

Bell inequalities

Aspect's experiment shows a violation of the Bell inequality

Non-local games



Game payoff : $$S = \sum_{x,y,a,b} V(a,b,x,y) \cdot \mathbb{P}(a,b|x,y)$$

Bell's inequality as a non-local game

- The type of answers Alice and Bob can give depend on their resources:
  - Classical strategies: with shared randomness $\lambda \sim \mu$, the players answer randomly given the local knowledge
  $$\mathbb{P}(a, b|x, y) = \int_\Lambda \mathbb{P}_A(a|x, \lambda) \cdot \mathbb{P}_B(b|y, \lambda) \, \mathrm{d}\mu(\lambda)$$
  - Quantum strategies: with a shared quantum bipartite state ψ, the players perform local POVMs
  $$\mathbb{P}(a, b|x, y) = \left\langle \psi \middle| A_x^a \otimes B_y^b \middle| \psi \right\rangle$$
- Correlation games: $V(a, b, x, y) = M_{xy} \cdot ab$ for some matrix $M \in \mathcal{M}_N(\mathbb{R})$.
- The CHSH game [CHSH69]: questions $\{x, y\} \in \{1, 2\}$, answers $\{a, b\} \in \{-1, 1\}$

$$S = \langle a_1 b_1 \rangle + \langle a_1 b_2 \rangle + \langle a_2 b_1 \rangle - \langle a_2 b_2 \rangle = \sum_{x,y=1}^{2} M_{x,y} \langle a_x, b_y \rangle$$

with $\langle a_x b_y \rangle = \sum_{a,b=\pm 1} ab \cdot \mathbb{P}(a, b|x, y)$ and $M_{\text{CHSH}} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

Bell's theorem [Bel64, Tsi87]: quantum theory violates the CHSH inequality

$$\sup\{S : \mathbb{P} \text{ quantum}\} = 2\sqrt{2} > 2 = \sup\{S : \mathbb{P} \text{ classical}\}$$

Norm characterization

Consider an N-input, 2-outcome game $M \in \mathcal{M}_N(\mathbb{R})$, and assume that

Alice's measurement operators $A = (A_1, \ldots, A_N)$ are fixed.

Definition

For a N-tuple of measurement operators on Alice's side $A = (A_1, \ldots, A_N)$, the largest quantum value of the game M defines a tensor norm

$$\|A\|_M := \sup_{\|\psi\|=1} \sup_{\|B_y\|\leq 1} \left\langle \psi \middle| \sum_{x,y=1}^{N} M_{xy} A_x \otimes B_y \middle| \psi \right\rangle = \lambda_{\max} \left[\sum_{y=1}^{N} \left| \sum_{x=1}^{N} M_{x,y} A_x \right| \right]$$

Alice's measurements are called M-Bell-local if for any choice of Bob's observables B and for any shared state ψ, one cannot violate the Bell inequality corresponding to M:

$$\sup\{S_M \ : \mathbb{P} \text{ quantum}\} =: \|A\|_M \leq \omega(M) := \sup\{S_M \ : \mathbb{P} \text{ classical}\}$$

Otherwise, they are called M-Bell-non-local.

Main results

Relating incompatibility to non-locality

It is known [WPGF09] that two measurements $A_1 = (A_1^+, A_1^-)$ and $A_2 = (A_2^+, A_2^-)$ violate the CHSH inequality if and only if they are incompatible.

We have introduced quantitative measures of non-locality and incompatibility: a N-tuple of dichotomic measurements $A = (A_1, \ldots, A_N)$ is

- compatible $\iff$ $\|A\|_c \leq 1$, with

$$\|A\|_c = \inf \left\{ \Big\| \sum_{j=1}^K H_j \Big\|_\infty \, : \, A = \sum_{j=1}^K z_j \otimes H_j, \text{ s.t. } \|z_j\|_\infty \leq 1 \text{ and } H_j \geq 0 \right\}$$

- not violating the Bell inequality M $\iff$ $\|A\|_M \leq \omega(M)$, with

$$\|A\|_M = \sup_{\|\psi\|=1} \sup_{\|B_y\|\leq 1} \left\langle \psi \middle| \sum_{x,y=1}^N M_{xy} A_x \otimes B_y \middle| \psi \right\rangle = \lambda_{\max} \left[\sum_{y=1}^N \left| \sum_{x=1}^N M_{x,y} A_x \right| \right]$$

Theorem ([Loulidi-Nechita, soon on the arXive])

For all invertible games M and measurements $A = (A_1, \ldots, A_N)$, it holds

$$\omega(M)^{-1} \cdot \|A\|_M \leq \|A\|_c \leq \|A\|_M \cdot \max \left\{ \left| (M^{-1})_{x,y} \right| \right\}_{x,y=1}^N$$

Relating incompatibility to Bell non-locality

- For the CHSH game ($N = 2$), the two norms are equal (for any Hilbert space dimension d):

$$\|A\|_c = \|A\|_{M_{\mathrm{CHSH}}}$$

- The above is a quantitative restatement of [WPGF09].
- How about other Bell inequalities in more general scenarios?

Theorem ([Loulidi-Nechita, soon on the arXive])

For all invertible non-local games $M \in \mathcal{M}_N(\mathbb{R})$, we have

$$\omega(M) \cdot \max \left\{ \left| (M^{-1})_{x,y} \right| \right\}_{x,y=1}^{N} \geq 1$$

with equality if and only if $N = 2$ and M is a permutation of M_{CHSH}.

In other words, the CHSH Bell inequality is essentially the only one which characterizes measurement incompatibility in the scenario where Alice's measurements are fixed.

Take home message

- Measurement compatibility (for dichotomic POVMs) can be characterized by a norm $\|\cdot\|_c$

$$(A_1, \ldots, A_N) \text{ compatible } \iff \|(A_1, \ldots, A_N)\|_c \leq 1$$

- For a N-input, 2-output non-local game M, the maximum value that can be obtained when Alice's measurements are fixed is given by the norm

$$\|A\|_M = \sup_{\|\psi\|=1} \sup_{\|B_y\|\leq 1} \left\langle \psi \middle| \sum_{x,y=1}^{N} M_{xy} A_x \otimes B_y \middle| \psi \right\rangle$$

- Bell inequality violations require incompatibility: $\|A\|_M \leq \|A\|_c \cdot \omega(M)$
- The reverse inequality holds, up to a constant: $\|A\|_c \leq \|A\|_M \cdot \max |(M^{-1})_{x,y}|$
- The CHSH Bell inequality (and its permutations) is the only one for which measurement incompatibility $\iff$ Bell non-locality:

$$\omega(M) \cdot \max |(M^{-1})_{x,y}| = 1 \implies M \cong M_{\text{CHSH}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Open question: measurements with ≥ 3 outcomes?

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