ENTANGLEMENT OF RANDOM QUANTUM STATES

Outline

1. Random Quantum States
2. Quantum Entanglement
3. The PPT criterion

References

arXiv > quant-ph > arXiv:1509.04689

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Quantum Physics

[Submitted on 15 Sep 2015]

Random matrix techniques in quantum information theory

Benoit Collins, Ion Nechita

The purpose of this review article is to present some of the latest developments using random techniques, and in particular, random matrix techniques in quantum information theory. Our review is a blend of a rather exhaustive review, combined with more detailed examples -- coming from research projects in which the authors were involved. We focus on two main topics, random quantum states and random quantum channels. We present results related to entropic quantities, entanglement of typical states, entanglement thresholds, the output set of quantum channels, and violations of the minimum output entropy of random channels.

arXiv > quant-ph > arXiv:1112.4582

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Quantum Physics

[Submitted on 20 Dec 2011 (v1), last revised 6 Apr 2012 (this version, v2)]

Phase transitions for random states and a semi-circle law for the partial transpose

Guillaume Aubrun, Stanislaw J. Szarek, Deping Ye

For a system of N identical particles in a random pure state, there is a threshold k_0 = k_0(N) ~ N/5 such that two subsystems of k particles each typically share entanglement if k > k_0, and typically do not share entanglement if k < k_0. By "random" we mean here "uniformly distributed on the sphere of the corresponding Hilbert space." The analogous phase transition for the positive partial transpose (PPT) property can be described even more precisely. For example, for N qubits the two subsystems of size k are typically in a PPT state if k < k_1 := N/4 - 1/2, and typically in a non-PPT state if k > k_1. Since, for a given state of the entire system, the induced state of a subsystem is given by the partial trace, the above facts can be rephrased as properties of random induced states. An important step in the analysis depends on identifying the asymptotic spectral density of the partial transposes of such random induced states, a result which is interesting in its own right.

① Random Quantum States

pure quantum states

$\mathcal{H}$ complex Hilbert space

$$\mathcal{H} = \mathbb{C}^d \qquad |\psi\rangle \in \mathbb{C}^d \qquad \|\psi\| = 1$$

Def random pure state = uniform point on the unit sphere of $\mathbb{C}^d$

In practice :

$\to$ Sample a random complex Gaussian vector

$$g \in \mathbb{C}^d \qquad g \sim \mathcal{N}(0, I_d)$$

(g_i) iid complex standard Gaussians

$\operatorname{Re} g_i$, $\operatorname{Im} g_i$ indep. have var. $\frac{1}{2}$

$$\mathbb{E}\,|g_i|^2 = 1$$

$$\mathbb{E}\,\|g\|^2 = d$$

$\to$ normalize it $\qquad |\psi\rangle := \dfrac{g}{\|g\|} \qquad \|\psi\| = 1$

uniform means $\quad |\psi\rangle \overset{(d)}{=} U \cdot |\psi\rangle \quad \forall\, U \in \mathcal{U}(d)$

mixed quantum states

(i.e. density matrices)

$$\mathcal{S} = \{ \rho \in \mathcal{M}_d^{sa}(\mathbb{C}) \;:\; \rho \geq 0 \quad \text{and} \quad \operatorname{Tr}\rho = 1 \}$$

$\uparrow$ positive semidefinite

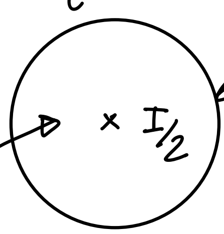
spectrum $(\rho) \subseteq [0, \infty)$

$\mathcal{S}$ is a convex set, with

$$\operatorname{ext}(\mathcal{S}) = \{ |\psi\rangle\langle\psi| \;:\; \psi \text{ pure state} \}$$

$d = 2$ (qubits)

$\mathcal{S}$ = Bloch ball



pure states
Bloch sphere

→ what is a random mixed quantum state?

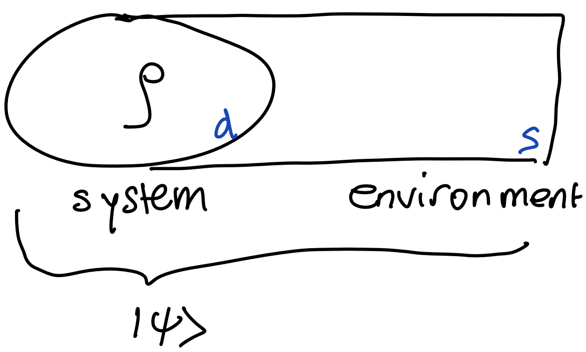
$\rho = U D U^*$ $\quad$ U = matrix of $\vec{ev}$

$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_d \end{pmatrix}$ diagonal matrix

→ choose U: random unitary rotation

→ distribution of D???

Theory of Open Quantum Systems



$|\psi\rangle$ pure state of global system (syst + env)

$\mathcal{H}_{system} = \mathbb{C}^d$

$\mathcal{H}_{env} = \mathbb{C}^s$

$\boxed{\rho = \mathrm{Tr}_{env} |\psi\rangle\langle\psi|}$

$|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^s = \mathbb{C}^{ds}$

Def If $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^s$ is uniform, we say that

$$\rho = [id_d \otimes \mathrm{Tr}_s](|\psi\rangle\langle\psi|) \sim \mu_{d,s}$$

$\mu_{d,s}$ = induced measure of params. (d,s)

$$\rho = \frac{\mathrm{Tr}_s |g\rangle\langle g|}{\|g\|^2} \qquad \left(\text{recall } |\psi\rangle = \frac{g}{\|g\|}\right)$$

$g \in \mathbb{C}^{ds} = \mathbb{C}^d \otimes \mathbb{C}^s$ $\qquad$ $g = (g_{ij})_{\substack{i=1\dots d \\ j=1\dots s}}$

$\quad \parallel$

$G \in \mathcal{M}_{d\times s}(\mathbb{C})$ $\qquad$ $(G)_{ij} = g_{ij}$

$$\mathbb{C}^{ds} = \mathbb{C}^d \otimes \mathbb{C}^s \simeq \mathbb{C}^d \otimes (\mathbb{C}^s)^* = \mathcal{M}_{d\times s}(\mathbb{C})$$

$$(\mathrm{Tr}_s |gXg|)_{ij} = \sum_{k=1}^{s} g_{ik} \overline{g_{jk}} = \sum_{k=1}^{s} G_{ik} (G^*)_{kj}$$
$$= (GG^*)_{ij}$$

partial trace = matrix multiplication

(special cases of a tensor contraction)

$$\rho = \frac{\mathrm{Tr}_s |gXg|}{\underbrace{\|g\|^2}_{\sum_{ij} |g_{ij}|^2 = \sum_{ij} |G_{ij}|^2 = \mathrm{Tr}(GG^*)}} = \frac{GG^*}{\mathrm{Tr}(GG^*)} = \frac{W}{\mathrm{Tr}\, W} \quad \text{with } W := GG^*$$

A random density matrix $\sim \mu_{d,s}$ ρ

$\rho = \frac{W}{\mathrm{Tr}\, W}$ with $W = G \cdot G^*$ where

$G \in \mathcal{M}_{d\times s}(\mathbb{C})$ with iid entries $G_{ij} \sim \mathcal{N}_{\mathbb{C}}(0,1)$

→ W is called a Wishart random matrix

→ $\mu_{d,s} \sim \rho$

size of ρ — is a parameter = size of the environment

→ pure states $|\psi\rangle \overset{(d)}{=} U \cdot |\psi\rangle \quad \forall U \in \mathcal{U}(d)$

→ mixed states $\rho \overset{(d)}{=} U \rho U^* \quad \forall U \in \mathcal{U}(d)$

$\rho \sim \mu_{d,s}$

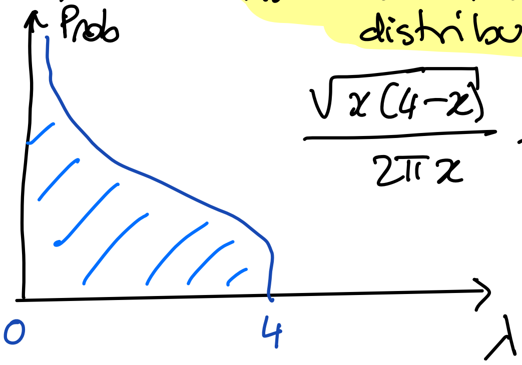
$\rho = V_\rho D_\rho V_\rho^*$ eigenvalue decomp

$V_\rho \sim$ uniform unitary matrix (Haar measure on $\mathcal{U}(d)$)

fact $\rho \sim \mu_{d,d}$ $(s = d)$

Marcenko - Pastur distribution

$$\mathrm{hist}\left(d \cdot \mathrm{eig}(\rho)\right) \xrightarrow[s=d\to\infty]{} \frac{\sqrt{x(4-x)}}{2\pi x}\, \mathbb{1}_{(0,4)}\, dx$$



$$\mathbb{E}\, \mathrm{Tr}(\rho^2) = \frac{2}{d}$$

max. possible value of S

$$\mathbb{E}\, S(\rho) \underset{d\to\infty}{\sim} \log d - \frac{1}{2}$$

$$-\mathrm{Tr}\, \rho \log \rho = -\sum_{i=1}^{d} \lambda_i \log \lambda_i$$

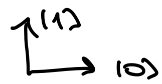
λ_i = eigs of ρ

② Quantum entanglement

pure states $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$

→ separable if $|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$

→ entangled otherwise



Examples $\mathbb{C}^2$ = qubits basis of $\mathbb{C}^2 = \{|0\rangle, |1\rangle\}$

sep: $|0\rangle \otimes |0\rangle = |00\rangle$

$|0\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$

entangled:

$$|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \neq |\varphi_1\rangle \otimes |\varphi_2\rangle$$

maximally entangled state

mixed states

Def A state $\rho \in \mathcal{M}_d \otimes \mathcal{M}_d$ is called <u>separable</u>

if $\rho = \sum_{i=1}^{r} p_i \, \underset{\substack{\vee\!\!| \\ 0}}{\alpha_i} \otimes \underset{\substack{\vee\!\!| \\ 0}}{\beta_i}$

where $p_1, \dots, p_r \geq 0$ $\sum p_i = 1$, α_i, β_i are q. states

Fact deciding whether a given state ρ is entangled or separable is NP-hard.

→ one looks at efficient entanglement criteria
separability

(" .... ") $\Rightarrow$ ρ entangled

PPT criterion

if $\rho \in SEP \iff \rho = \sum p_i \alpha_i \otimes \beta_i$

$$\Rightarrow \underbrace{[\mathrm{id} \otimes \mathrm{transp}](\rho)}_{\substack{\text{partial transp. of } \rho \\ \rho^{\Gamma}}} = \sum p_i \underbrace{\alpha_i}_{\geq 0} \otimes \underbrace{\beta_i^T}_{\geq 0} \geq 0$$

So: $\rho \in SEP \Rightarrow \rho^{\Gamma} \geq 0$

$$\underbrace{\rho^{\Gamma} \not\geq 0}_{\rho^{\Gamma} \text{ has negative eigenvalues}} \Rightarrow \rho \text{ entangled}$$

Example

$$\underset{\substack{\Uparrow \\ \mathbb{C}^2}}{|\Omega\rangle} = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right) = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{array} \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$$

$\omega = |\Omega\rangle\langle\Omega|$

$$\omega = \frac{1}{2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right] \in \mathcal{M}_4 \simeq \mathcal{M}_2 \otimes \mathcal{M}_2$$

$$\omega^\Gamma = (\mathrm{id} \otimes \mathrm{transp})(\omega) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [1] \oplus [1] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has <0 eigs

$\to \omega^\Gamma \not\geq 0 \Rightarrow \omega$ is entangled

③ PPT criterion for random density matrices

Theorem I (Aubrun, Szarek, Ye)

$\rho \sim \mu_{d^2,s}$

A | B | env — d, d, s

if $s \gtrsim d^3 \log^2 d$ $\qquad \mathbb{P}(\rho \in SEP) \xrightarrow{d\to\infty} 1$

if $s \lesssim d^3$ $\qquad \mathbb{P}(\rho \in SEP) \xrightarrow[d\to\infty]{} 0$

Theorem II $\qquad \rho \in \mu_{d^2,s} \qquad s \sim c d^2$

if $c > 4$ $\qquad \mathbb{P}(\rho^\Gamma \geq 0) \xrightarrow{d\to\infty} 1$

if $c < 4$ $\qquad \mathbb{P}(\underbrace{\rho^\Gamma \not\geq 0}_{\Rightarrow\ \rho \notin SEP}) \xrightarrow[d\to\infty]{} 1$

$\rho \geq 0 \qquad \mathrm{Tr}\,\rho = 1$

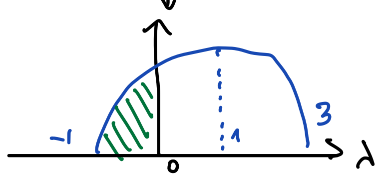
$\rho^\Gamma \not\geq 0 \qquad \mathrm{Tr}\,\rho^\Gamma = 1$

$\|\rho^\Gamma\|_1 = \sum |\lambda_i(\rho^\Gamma)| \geq 1$

eq $\iff \rho^\Gamma \geq 0$

$$\mathrm{list}(d^2 \cdot \mathrm{eig}(\rho^\Gamma))$$

$\Big\downarrow d \to \infty \qquad (s = d), \ c = 1$



Q: How to measure the non PPT-ness of ρ?