Diagonal Unitary and Orthogonal Symmetries in Quantum Theory

joint work with Satvik Singh (Cambridge UK)

arXiv:2010.07898 and arXiv:2112.11123

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Plan of the talk

1. Unitary symmetry in quantum information - isotropic states
2. Diagonal unitary / orthogonal symmetry
3. Positive semidefinite operators
4. Unitary operators
5. Dual unitary matrices, entangling power, and Hadamard matrices

Unitary symmetry in quantum information

Symmetry in quantum theory

- Quantum states are modeled mathematically by density matrices

$$\{\rho \in \mathscr{M}_d(\mathbb{C}) \,:\, \rho \geq 0 \text{ and } \operatorname{Tr}\rho = 1\}$$

- Unitary operators encode time evolution

$$\rho \mapsto U\rho U^*, \quad \text{for } U \in \mathscr{U}_d$$

- We would like to consider quantum states which are left invariant by a certain class of unitary operators

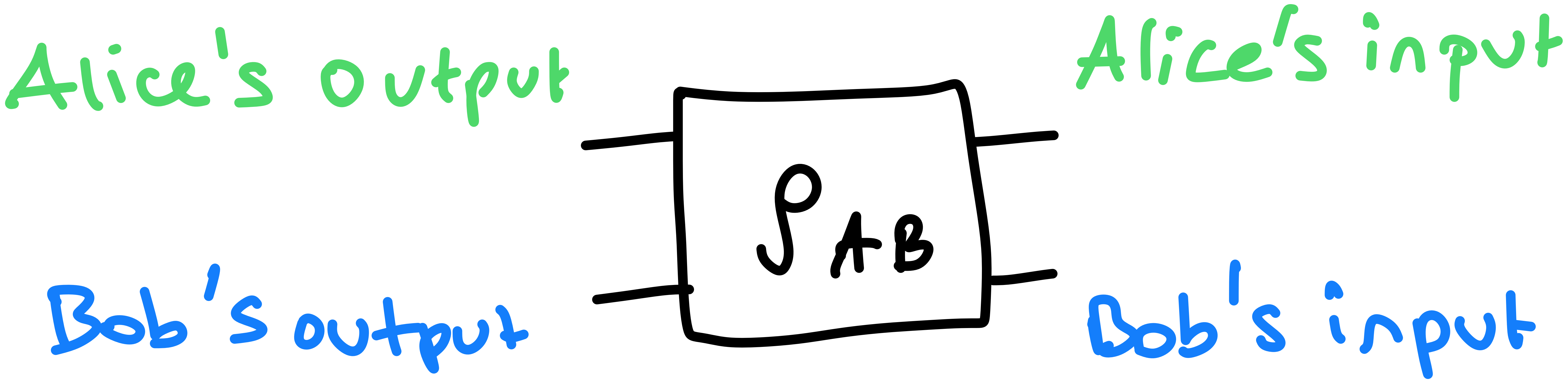
Example. If $UXU^* = X$ for all $U \in \mathscr{U}_d$, then X must be a scalar matrix, i.e. $X = cI_d$ for some constant $c \in \mathbb{C}$. For density matrices, $c = 1/d$.

Bipartite operators

- Quantum entanglement is of the key features of quantum theory. It appears when one considers two quantum systems.

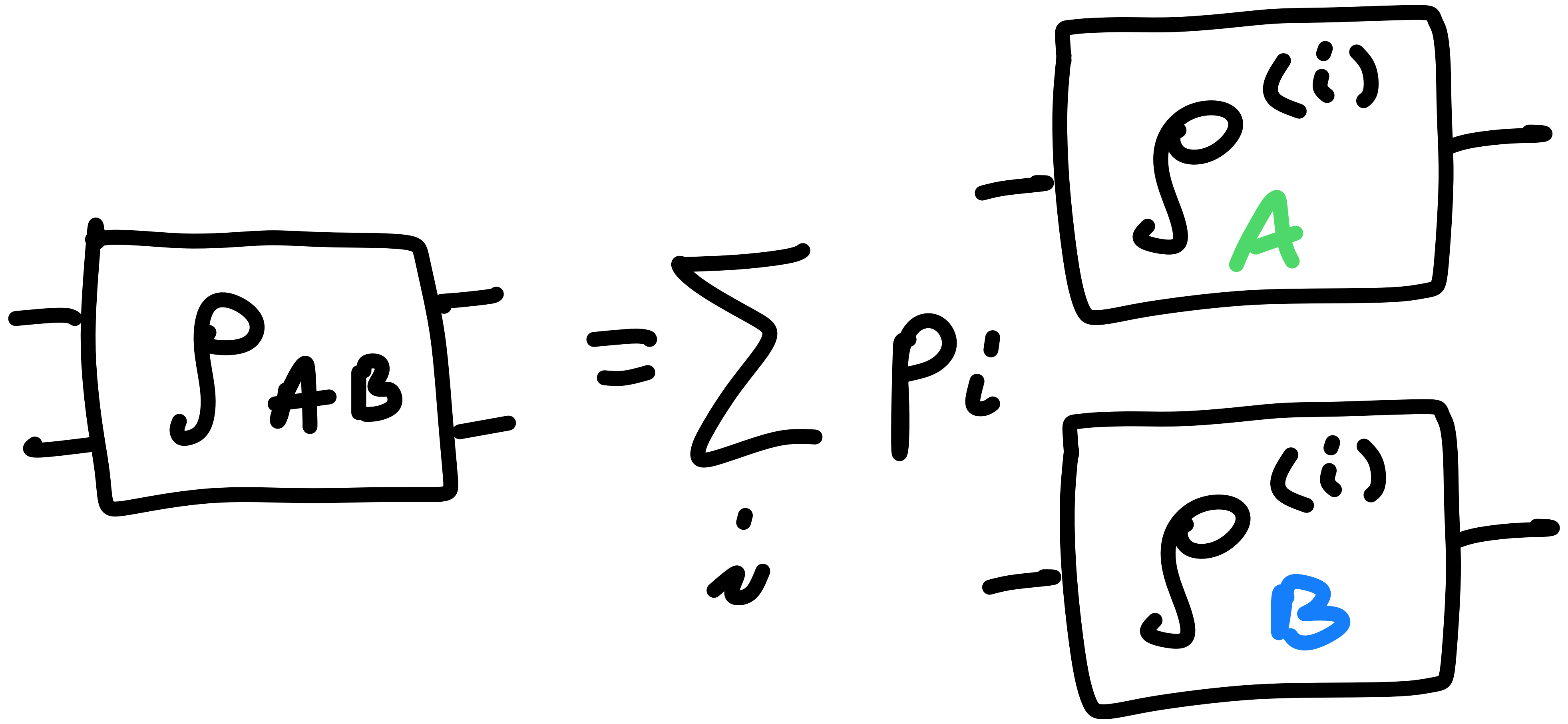
- In this case, the set of quantum states is given by bipartite matrices:

$$\{\rho_{AB} \in \mathscr{M}_d \otimes \mathscr{M}_d \cong \mathscr{M}_{d^2} \ : \ \rho \geq 0 \text{ and } \operatorname{Tr}\rho = 1\}$$



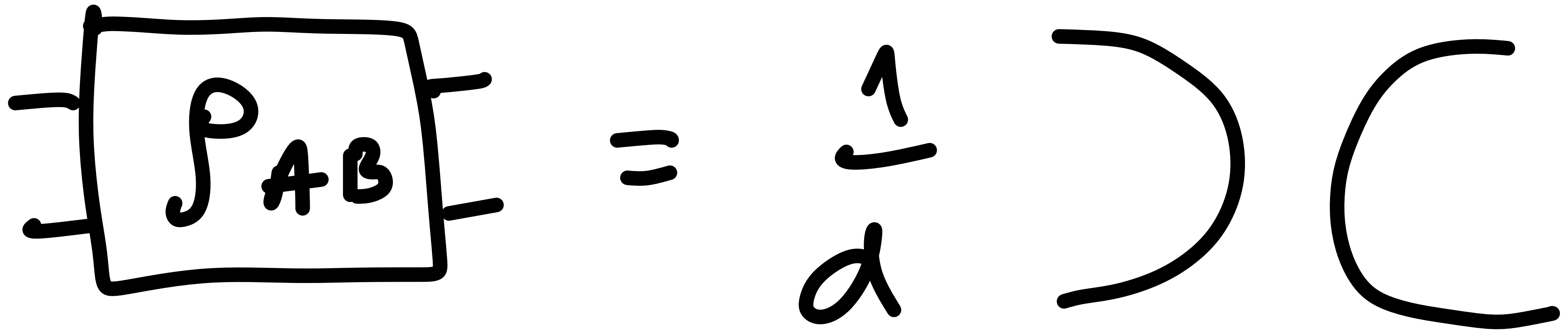
Two examples

- Separable states are bipartite quantum states which can be written as convex combinations of product states $\rho_A \otimes \rho_B$



- Entangled states are the non-separable states. The most important example is the maximally entangled state

$$\omega = \frac{1}{d} \sum_{i,j=1}^{d} |ii\rangle\langle jj|$$

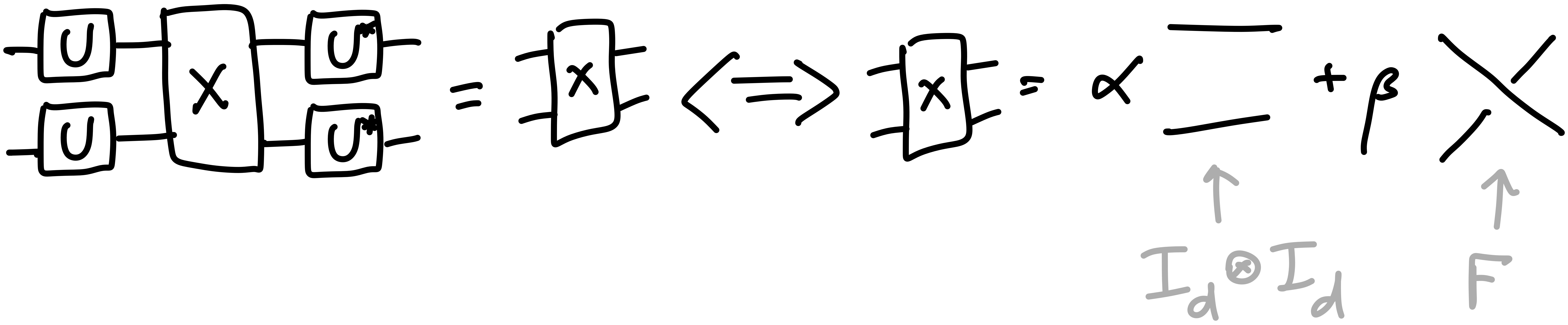


(Full) Unitary symmetry in the bipartite case

Theorem. Let $X \in \mathscr{M}_d \otimes \mathscr{M}_d$ be a bipartite operator. Then

$$\forall U \in \mathscr{U}_d, (U \otimes U)X(U \otimes U)^* = X \iff X = \alpha I_{d^2} + \beta F \text{ for } \alpha, \beta \in \mathbb{C}$$

where the (unitary) flip operator is defined by $F\, x \otimes y = y \otimes x$.

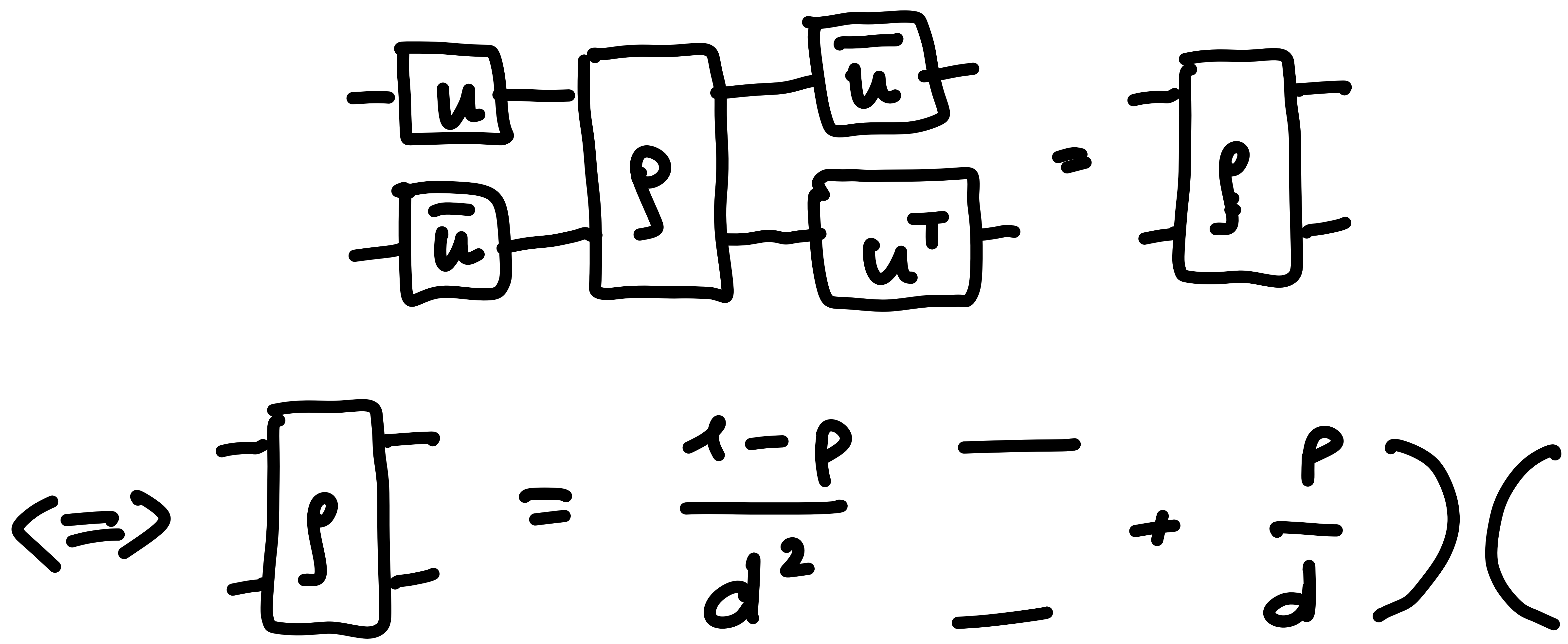


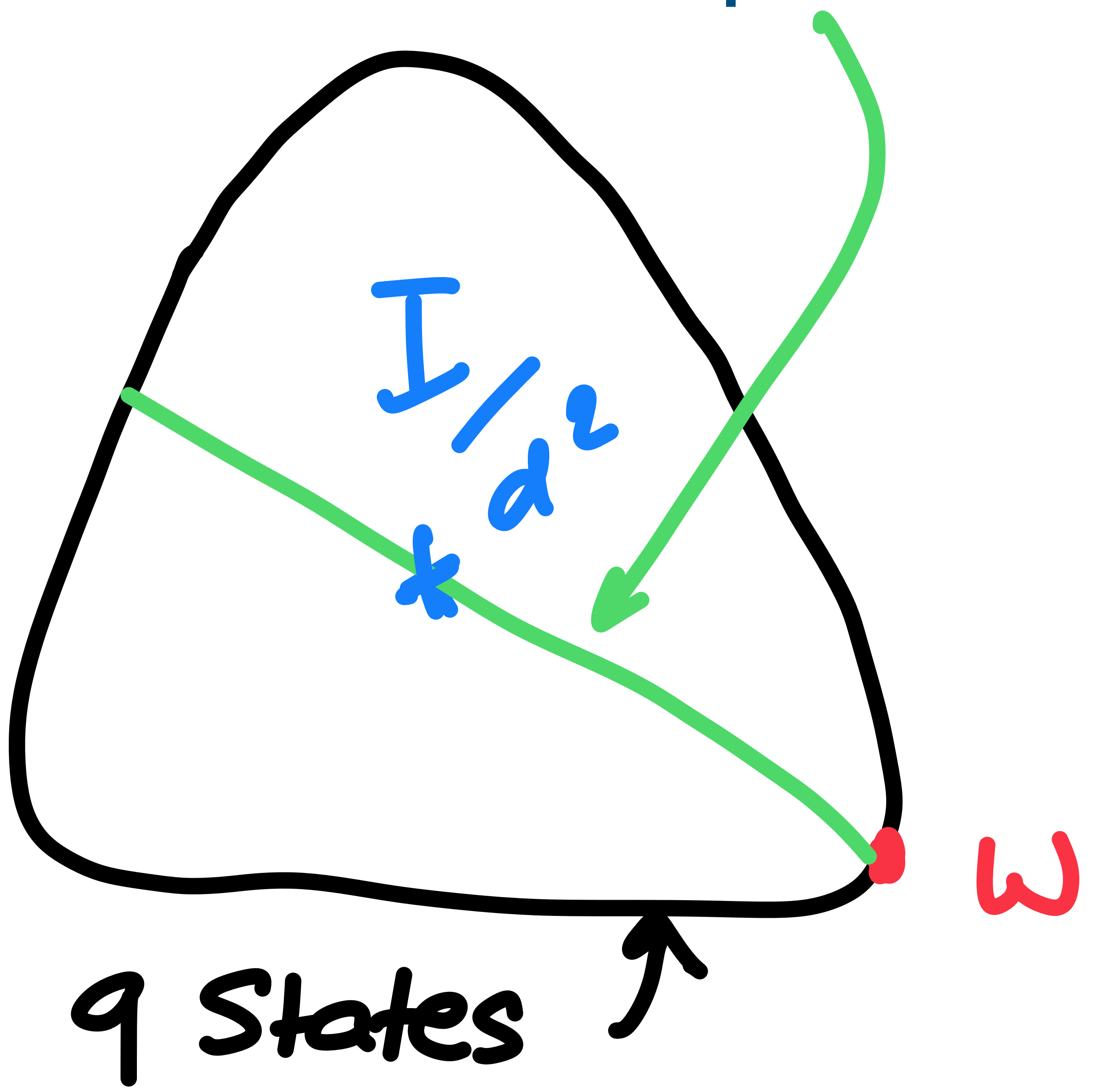
Isotropic quantum states

Theorem. Let $\rho \in \mathscr{M}_d \otimes \mathscr{M}_d$ be a bipartite density matrix. Then

$$\forall U \in \mathscr{U}_d, (U \otimes \bar{U})\rho U^* \otimes U^\top = \rho \iff \rho = (1-p)\frac{I}{d^2} + p\omega \text{ for } p \in [-1/(d^2-1),1]$$

i.e. ρ must be a convex combination of the maximally mixed state and the maximally entangled state. Such quantum states are called isotropic.





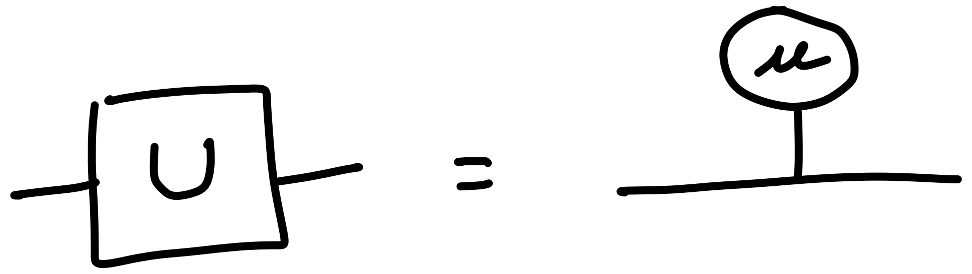
Diagonal unitary/orthogonal symmetry

The diagonal subgroup

- Since requiring the full unitary symmetry yields matrices (resp. quantum states) with only 2 parameters, we shall consider the much smaller subgroups

$$\mathscr{D}\mathscr{U}_d := \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_d}) \ : \ \theta \in \mathbb{R}^d\}$$

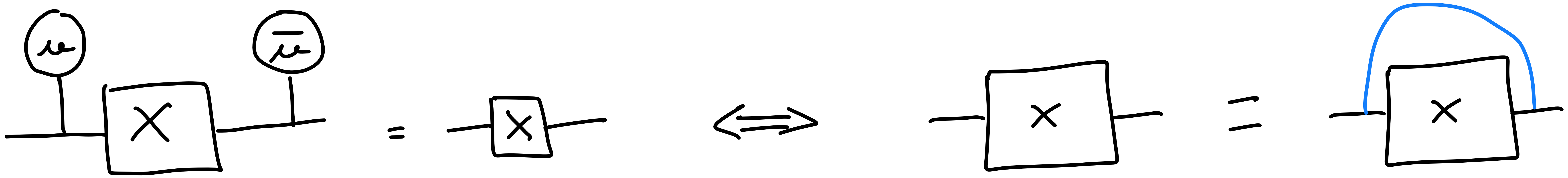
$$\mathscr{D}\mathscr{O}_d := \{\mathrm{diag}(\epsilon_1, \ldots, \epsilon_d) \ : \ \epsilon \in \{\pm 1\}^d\}$$



$$u_{ij} = \delta_{ij} \cdot \mu_i$$

- In the case of a single tensor factor, we have

$$\forall U \in \mathscr{D}\mathscr{U}_d, \quad UXU^* = X \iff X = \mathrm{diag}(X)$$

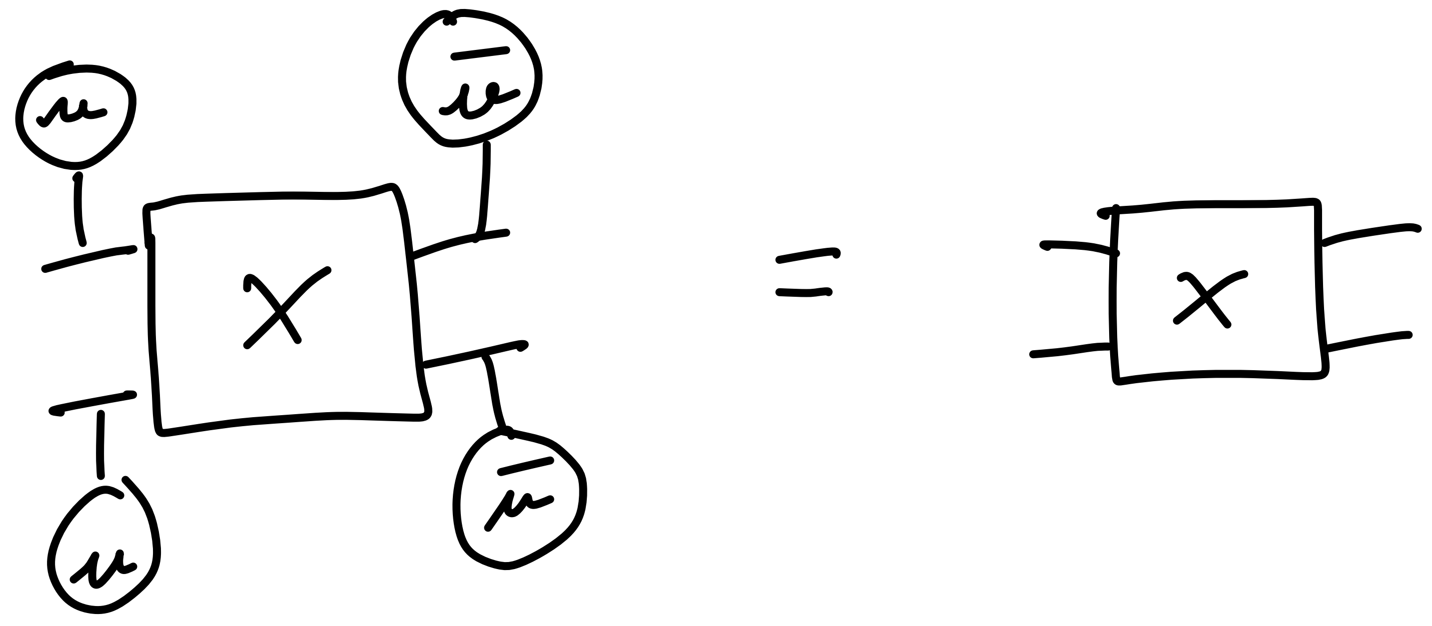


Diagonally symmetric bipartite matrices

Definition. A bipartite matrix $X \in \mathscr{M}_d \otimes \mathscr{M}_d$ is called:

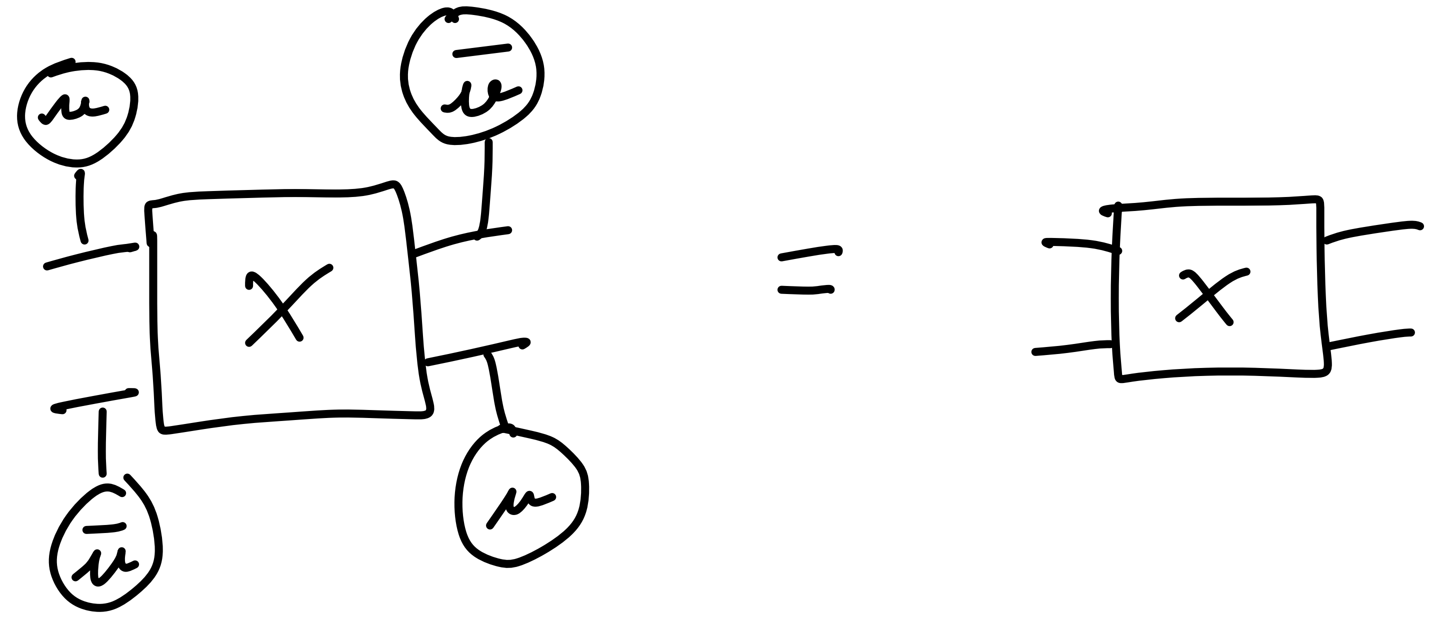
- LDUI (local diagonal unitary invariant) if

$$\forall U \in \mathscr{DU}_d, \quad (U \otimes U)X(U^* \otimes U^*) = X$$



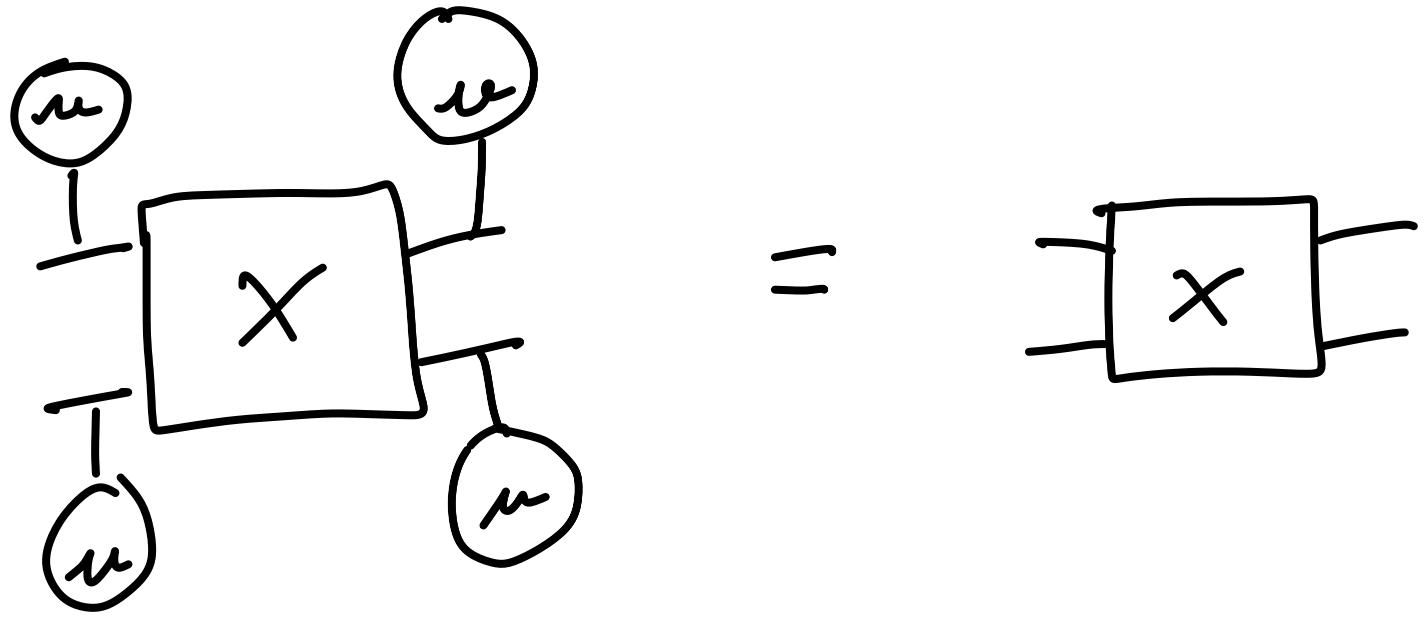
- CLDUI (conjugate LDUI) if

$$\forall U \in \mathscr{DU}_d, \quad (U \otimes \bar{U})X(U^* \otimes U^\top) = X$$



- LDOI (local diagonal orthogonal invariant) if

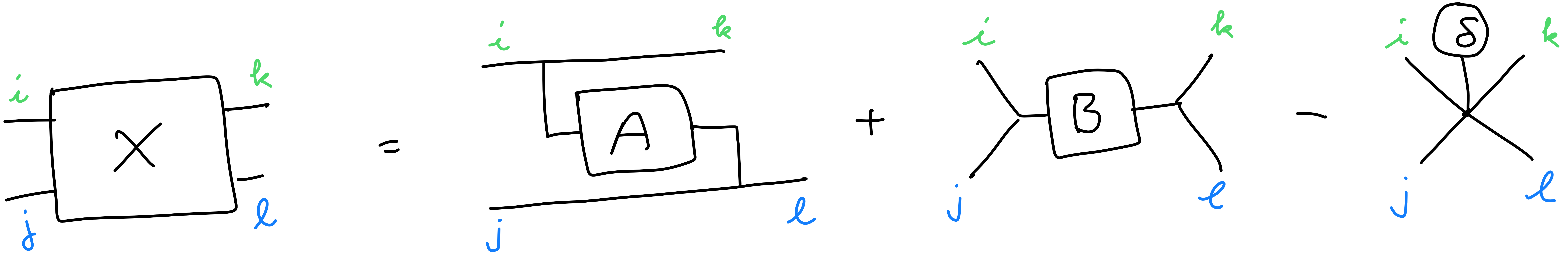
$$\forall U \in \mathscr{DO}_d, \quad (U \otimes U)X(U^\top \otimes U^\top) = X$$

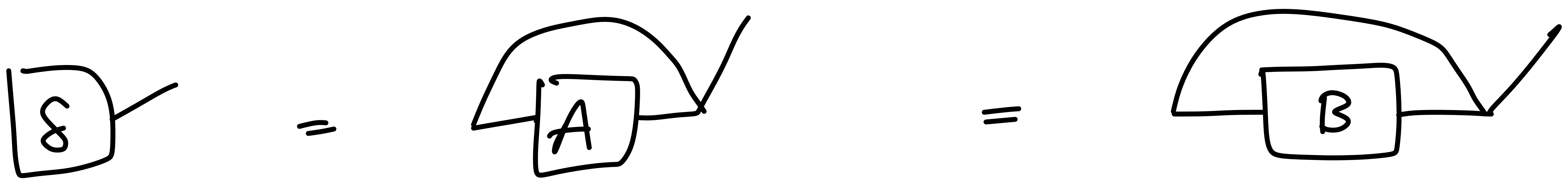


Characterization theorem - CDLUI case

Theorem. A matrix $X \in \mathscr{M}_d \otimes \mathscr{M}_d$ is CLDUI iff there exist matrices $A, B \in \mathscr{M}_d$ having the same diagonal $\operatorname{diag} A = \operatorname{diag} B =: \delta \in \mathbb{C}^d$ such that

$$X_{ij,kl} = \mathbf{1}_{i=k,j=l} A_{ij} + \mathbf{1}_{i=j,k=l} B_{ik} - \mathbf{1}_{i=j=k=l} \delta_i$$





Characterization theorem - LDUI and LDOI

Theorem. A matrix $X \in \mathscr{M}_d \otimes \mathscr{M}_d$ is LDUI iff there exist matrices $A, C \in \mathscr{M}_d$ having the same diagonal $\operatorname{diag} A = \operatorname{diag} C =: \delta \in \mathbb{C}^d$ such that

$$X_{ij,kl} = \mathbf{1}_{i=k,j=l} A_{ij} + \mathbf{1}_{i=l,j=k} C_{ij} - \mathbf{1}_{i=j=k=l} \delta_i$$

Theorem. A matrix $X \in \mathscr{M}_d \otimes \mathscr{M}_d$ is LDOI iff there exist matrices $A, B, C \in \mathscr{M}_d$ having the same diagonal $\operatorname{diag} A = \operatorname{diag} B = \operatorname{diag} C =: \delta \in \mathbb{C}^d$ such that

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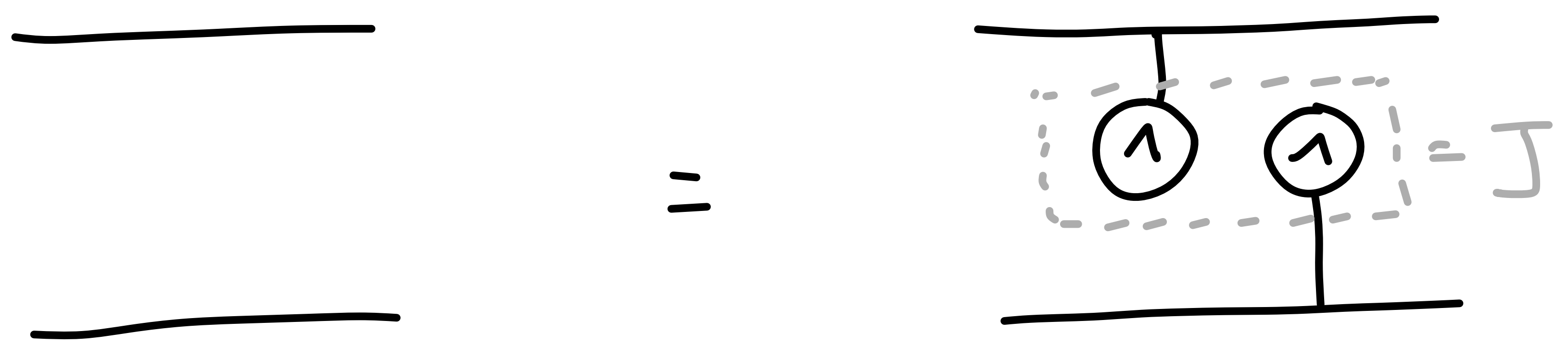
Three examples

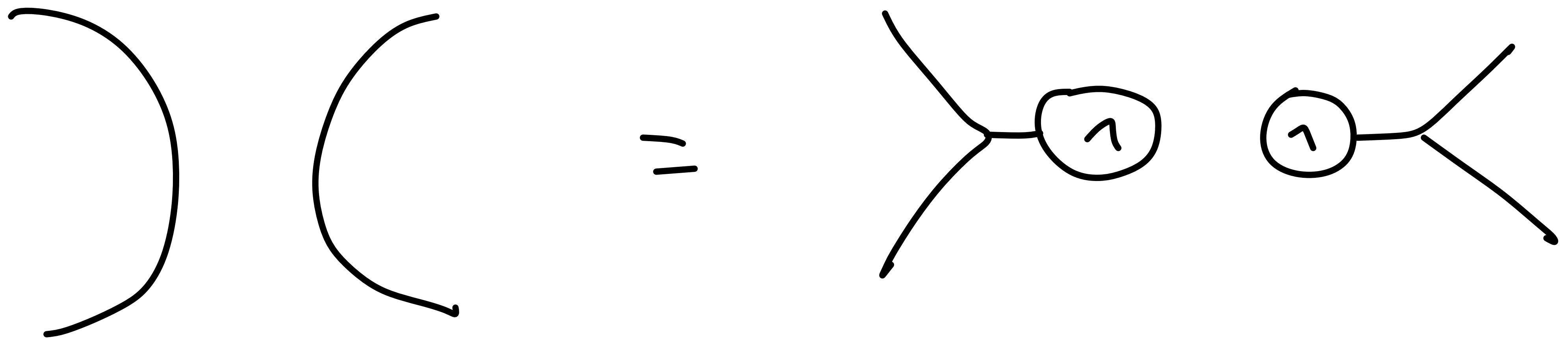
- The identity matrix is CLDUI with

  $A = J_d, B = I_d$



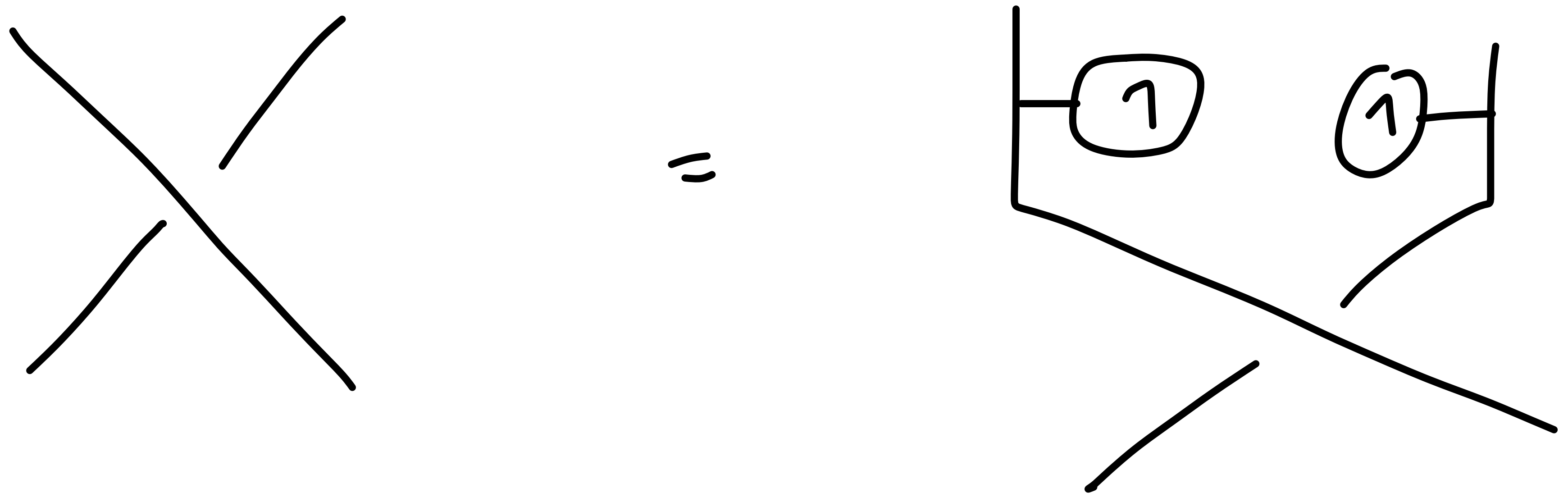
- The maximally entangled state is

  CLDUI with $A = I_d, B = J_d$



- The flip operator is LDUI with

  $A = I_d, B = J_d$



Symmetric bipartite PSD operators

Properties of symmetric operators

Theorem. A bipartite LDOI operator $X = X_{A,B,C}$ is

- self-adjoint iff A is real and B, C are self-adjoint
- positive semidefinite iff the following three conditions hold:
    1. $A_{ij} \geq 0$ for all i, j
    2. B is positive semidefinite
    3. $A_{ij}A_{ji} \geq |C_{ij}|^2$ for all i, j

Note that LDUI operators correspond to B diagonal, and CLDUI operators correspond to C diagonal.

Separability for diagoal symmetric matrices

Theorem. A bipartite CLDUI operator of the form $X = X_{A,A}$ is separable iff the matrix A is completely positive, i.e.

$$\exists R \in \mathscr{M}_{d\times D}(\mathbb{R}_+) \quad \text{s.t.} \quad A = RR^\top$$

In dimensions $d \leq 4$ completely positive matrices are precisely positive semidefinite matrices with non-negative entries.

Appropriate generalizations of the cone of completely positive matrices exist to characterize separability of arbitrary CLDUI matrices $X_{A,B}$ (pairwise completely positive matrices) and of LDOI matrices $X_{A,B,C}$ (triplewise completely positive matrices). The membership problem for these cones is still NP-hard.

Symmetric bipartite unitary operators

Unitary diagonal symmetric operators

LDOI bipartite operators $X = X_{A,B,C}$ admit the following block-decomposition

$$X = B \oplus \left(\bigoplus_{i<j} \begin{bmatrix} A_{ij} & C_{ij} \\ C_{ji} & A_{ji} \end{bmatrix} \right)$$

Theorem. A LDOI matrix $X = X_{A,B,C}$ is unitary iff the following conditions hold

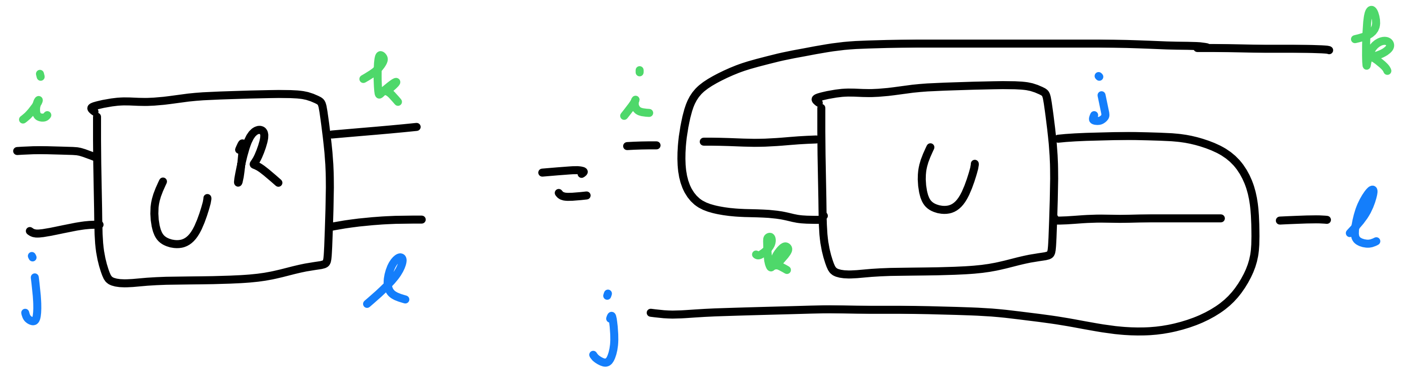
1. B is unitary
2. $\forall i < j,$ there exists a phase ω_{ij} such that $A_{ji} = \omega_{ij} \bar{A}_{ij}$ and $A_{ji} = -\omega_{ij} \bar{C}_{ij}$
3. $\forall i < j \quad |A_{ij}|^2 + |C_{ij}|^2 = 1$

@#!*& and Hadamard matrices

Dual unitaries

- A bipartite unitary matrix $X \in \mathcal{U}_{d^2}$ is called a dual unitary if its realignment U^R is also a unitary matrix

$$U^R_{ij,kl} = U_{ik,jl}$$

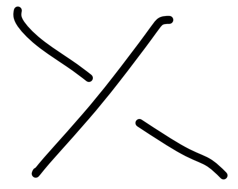


Theorem. An LDOI matrix $X_{A,B,C}$ is dual unitary iff

1. A and B are unitary
2. $\forall i < j \quad |A_{ij}|^2 = |B_{ij}|^2 = 1 - |C_{ij}|^2$
3. $\forall i < j$, there exist complex phases $\omega_{ij} \in \mathbb{T}$ such that

$$A_{ji} = \omega_{ij}\overline{A_{ij}} \qquad B_{ji} = \omega_{ij}\overline{B_{ij}} \qquad C_{ji} = -\omega_{ij}\overline{C_{ij}}$$

Example. $F^R = F$



Example. $I^R = d\omega$

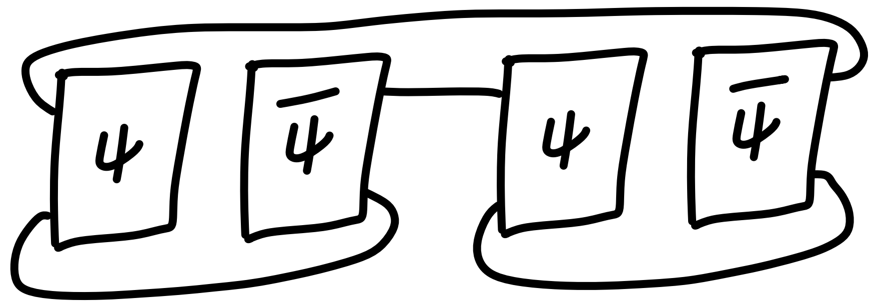
In particular, an LDUI matrix $X_{A,C}$ is dual unitary iff

$$\forall i, j \quad C_{ij} \in \mathbb{T} \quad \text{and} \quad A = \text{diag}(C)$$

Entangling power of a bipartite unitary

- Given a bipartite pure quantum state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, we define its linear entropy as

$$\mathscr{E}(|\psi\rangle) = 1 - \mathrm{Tr}[(\mathrm{Tr}_2|\psi\rangle\langle\psi|)^2] = 1 -$$



- The linear entropy is an entanglement measure: it measures how entangled the pure state $|\psi\rangle$ is

- For a bipartite unitary matrix $X \in \mathscr{U}_{d^2}$, we define its entangling power as the amount of entanglement it creates, on average, when acting on a separable state

$$e_p(X) = \left(\frac{d+1}{d-1}\right) \mathbb{E}_{\phi,\psi\sim\mathrm{Haar}}[\mathscr{E}(X \cdot |\phi\rangle \otimes |\psi\rangle)],$$

Maximum entangling power and Hadamard matrices

- Theorem. The maximum entangling power of an LDUI dual unitary matrix $X_{A,C}$ is achieved when C is a complex Hadamard matrix, with the maximum value

$$\max_{X \text{ dual unitary}} e_p(X) = \frac{d}{d+1} \to 1 \quad \text{as} \quad d \to \infty \,.$$

- In this case, the quantity $e_p(X)$ is related to the following measure of Hadamardness

$$\mathfrak{h}(C) := \mathrm{Tr}[(CC^\dagger)^2] = \sum_{i,j=1}^{d} |\langle C_{i,\cdot}, C_{j,\cdot}\rangle|^2 = \sum_{i,j=1}^{d} \left| \sum_{k=1}^{d} C_{ik}\bar{C}_{jk} \right|^2$$

Real Hadamards

In the case where the matrix C is real, the quantity $\mathfrak{h}(C)$ cannot attain its minimum value d^3 unless d is a multiple of 4

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$\min \mathfrak{h}(C)$ for sign matrices C			
Matrix size	$\min \mathfrak{h}(C)$	#argmin $\mathfrak{h}$	$C \in \operatorname{argmin} \mathfrak{h}$
$d=3$	33	6	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$
$d=5$	145	120	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{bmatrix}$
$d=6$	264	28800	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}$

Open question. What is the minimal value of $\mathfrak{h}(C)$ for values of d which are not multiples of 4? Is it true that for odd values of d, it is attained for matrices C with the property that all the scalar products between different rows have absolute value 1?

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