Some applications of free spectrahedra to quantum information theory

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What this talk is about

We establish a connection between two topics:

- Incompatibility of quantum measurements
- Inclusion of free spectrahedra, geometric objects which generalize polyhedra

The relation we put forward allows us to find the most incompatible quantum measurements in different settings. These objects are of great importance for the foundations of quantum mechanics due to their non-classical behavior. They also encode, in some sense, the limits of quantum theory.

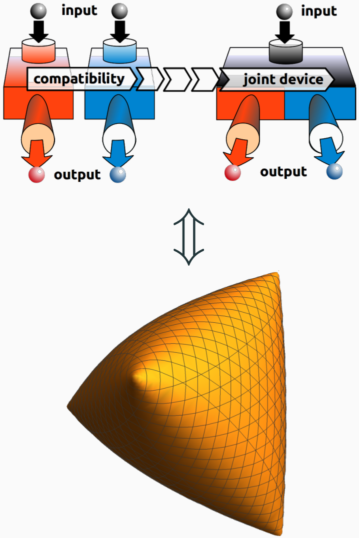
Talk outline

Incompatibility in QM

Free spectrahedra

Main results

Proof ideas



Incompatibility in QM

Quantum measurements. Compatibility

- Quantum states $\rightsquigarrow$ unit vectors in a Hilbert space: $\psi \in \mathbb{C}^d$, $\|\psi\| = 1$
- Quantum measurements are modeled by POVMs (Positive Operator Valued Measures)

$$\boxed{A_1, \ldots, A_k \in M_d(\mathbb{C})^{\mathrm{sa}}, \quad A_i \geq 0, \quad \sum_{i=1}^{k} A_i = I_d}$$

- Born's rule: $\mathbb{P}[\text{outcome } i \text{ is observed}] = \langle \psi, A_i \psi \rangle$

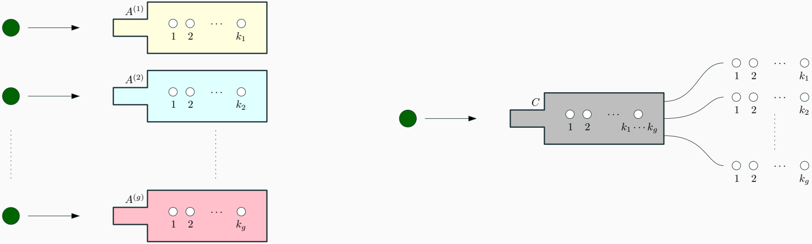
Definition

Two POVMs, $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_l)$, are called compatible if there exists a third POVM $C = (C_{ij})_{i \in [k], j \in [l]}$ such that

$$\forall i \in [k], \quad A_i = \sum_{j=1}^{l} C_{ij} \qquad \text{and} \qquad \forall j \in [l], \quad B_j = \sum_{i=1}^{k} C_{ij}$$

The definition generalizes to g-tuples of POVMs $A^{(1)}, \ldots, A^{(g)}$, having respectively $k_1, \ldots k_g$ outcomes, where the joint POVM C has outcome set $[k_1] \times \cdots \times [k_g]$.

What does it mean?



- Compatible measurements can be simulated by a single joint measurement, by classically post-processing its outputs
- Examples:
  1. Trivial POVMs $A = (p_i I_d)$ and $B = (q_j I_d)$ are compatible
  2. Commuting POVMs $[A_i, B_j] = 0$ are compatible
  3. If the POVM A is projective, then A and B are compatible iff they commute
- Compatibility appears also in the setting of quantum channels [HMZ16]

Noisy POVMs

- POVMs can be made compatible by adding noise, i.e. mixing in trivial POVMs
- Example: dichotomic POVMs and white noise, $s \in [0,1]$

$$(E, I-E) \mapsto s(E, I-E) + (1-s)(\frac{I}{2}, \frac{I}{2}) \qquad \text{or} \qquad E \mapsto sE + (1-s)\frac{I}{2}$$

- Taking $s = 1/2$ suffices to render any pair of dichotomic POVMs compatible [DFK19]

  $\rightsquigarrow$ define $C_{ij} := (E_i + F_j)/4$

- From now on, we focus on dichotomic (YES/NO) POVMs

Definition

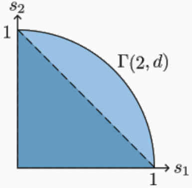
The compatibility region for g measurements on $\mathbb{C}^d$ is the set

$$\Gamma(g,d) := \{s \in [0,1]^g \ : \ \text{for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}), \\ \text{the noisy versions } s_i E_i + (1-s_i) I_d/2 \text{ are compatible}\}$$

Compatibility region

$$\Gamma(g,d) := \{s \in [0,1]^g \ : \text{ for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}),$$
$$\text{the noisy versions } s_i E_i + (1 - s_i) I_d/2 \text{ are compatible}\}$$

- The set $\Gamma(g,d)$ is convex
- For all $i \in [g]$, $e_i \in \Gamma(g,d)$: every measurement is compatible with $g-1$ trivial measurements
- For $d \geq 2$, $(1,1,\ldots,1) \notin \Gamma(g,d)$: there exist incompatible measurements
- For all $d \geq 2$, $\Gamma(2,d)$ is a quarter-circle [BH08, BHSS13]
- Generally speaking, the set $\Gamma(g,d)$ tells us how robust (to noise) is the incompatibility of g dichotomic measurements on $\mathbb{C}^d$



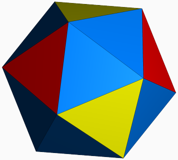
GOAL: compute the set $\Gamma(g,d)$ for all g,d

Free spectrahedra

Free spectrahedra

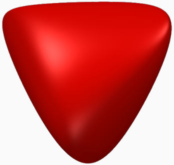
- A polyhedron is defined as the intersection of half-spaces

$$\{x \in \mathbb{R}^g \,:\, \langle h_i, x\rangle \leq 1, \quad \forall i \in [k]\}$$



- A spectrahedron is given by PSD constraints: for $A = (A_1, \dots, A_g) \in (\mathcal{M}_d^{sa})^g$

$$\mathcal{D}_A(1) := \{x \in \mathbb{R}^g \,:\, \sum_{i=1}^{g} x_i A_i \leq I_d\}$$



- $\mathcal{D}_{(\sigma_X, \sigma_Y, \sigma_Z)}(1) = \{(x, y, z) \in \mathbb{R}^3 \,:\, x\sigma_X + y\sigma_Y + z\sigma_Z \leq I_2\} = \text{Bloch ball}$
- A free spectrahedron is the matricization of a spectrahedron [Vin14]

$$\mathcal{D}_A := \coprod_{n=1}^{\infty} \mathcal{D}_A(n) \quad \text{with} \quad \mathcal{D}_A(n) := \{X \in (\mathcal{M}_n^{sa})^g \,:\, \sum_{i=1}^{g} X_i \otimes A_i \leq I_{nd}\}$$

Examples: the cube and the diamond

The matrix cube is the free spectrahedron defined by

$$\mathcal{D}_{\square,g} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \|X_i\| \leq 1, \quad \forall i \in [g]\}$$

- At level one, $\mathcal{D}_{\square,g}(1)$ is the unit ball of the ℓ^∞ norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2g \times 2g$ diagonal matrices $\mathcal{D}_{\square,g} = \mathcal{D}_{K_1,\ldots,K_g}$, with $K_i = \mathrm{diag}(e_i) \oplus \mathrm{diag}(-e_i)$

The matrix diamond is the free spectrahedron defined by

$$\mathcal{D}_{\diamondsuit,g} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^{g} \varepsilon_i X_i \leq I_n, \quad \forall \varepsilon \in \{\pm 1\}^g\}$$

- At level one, $\mathcal{D}_{\diamondsuit,g}(1)$ is the unit ball of the ℓ^1 norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2^g \times 2^g$ diagonal matrices $\mathcal{D}_{\diamondsuit,g} = \mathcal{D}_{L_1,\ldots,L_g}$, with $L_i = I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2$

Spectrahedral inclusion

- Consider two free spectrahedra defined by g-tuples of matrices $(A_1, \ldots, A_g)$ and $(B_1, \ldots, B_g)$
- We say that $\mathcal{D}_A$ is contained in $\mathcal{D}_B$ and we write $\mathcal{D}_A \subseteq \mathcal{D}_B$ if, for all $n \geq 1$, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$
- Clearly, $\mathcal{D}_A \subseteq \mathcal{D}_B \implies \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$. For the converse implication to hold, one may need to shrink $\mathcal{D}_A$...

Definition

For a free spectrahedron $\mathcal{D}_A$, we define its set of inclusion constants as

$$\Delta_A(g, d) := \{s \in [0,1]^g \ : \text{for all } g\text{-tuples } B_1, \ldots, B_g \in \mathcal{M}_d(\mathbb{C})^{\mathrm{sa}}, \\ \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s.\mathcal{D}_A \subseteq \mathcal{D}_B\}$$

- The inclusion constants for the matrix cube play an important role in combinatorial optimization [HKMS19]
- We shall be concerned with the inclusion set for the matrix diamond, which we denote by $\Delta(g, d)$; it is the same as the one of the matrix cube [BN22a]

Main results

Compatibility in QM $\iff$ matrix diamond inclusion

To a g-tuple of selfadjoint matrices $E \in (\mathcal{M}_d^{sa})^g$, we associate the free spectrahedron defined by the matrices $2E_i - I_d$:

$$\mathcal{D}_{2E-I} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^{g} X_i \otimes (2E_i - I_d) \leq I_{nd}\}$$

Theorem ([BN18])

Let $E \in (\mathcal{M}_d^{sa})^g$ be g-tuple of selfadjoint matrices. Then:

- *The matrices E are quantum effects* $\iff$ $\mathcal{D}_{\diamondsuit,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$
- *The matrices E are compatible quantum effects* $\iff$ $\mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$

At the intermediate levels $1 \leq n \leq d$, $\mathcal{D}_{\diamondsuit,g}(n) \subseteq \mathcal{D}_{2E-I}(n)$ iff for all isometries $V : \mathbb{C}^n \to \mathbb{C}^d$, the compressed effects $V^* E_i V$ are compatible.

Moreover, the compatibility region is equal to the set of inclusion constants of the matrix diamond: $\forall g, d$, $\Gamma(g, d) = \Delta(g, d)$.

The same results hold in the general (non-dichotomic) setting. One has to replace the matrix diamond by the matrix jewel $\mathcal{D}_{\text{jewel},\mathbf{k}}$ [BN20]

Consequences

Many things are known about the matrix diamond

- For all g, d, $\frac{1}{2d}(1, 1, \ldots, 1) \in \Delta(g, d)$ [HKMS19]
- For all g, d, $\mathrm{QC}_g := \{s \in [0,1]^g \,:\, \sum_i s_i^2 \leq 1\} \subseteq \Delta(g, d)$ [PSS18]

Many things are known about (in-)compatibility

- Many small g, d cases completely solved
- Approximate quanutum cloning [Kay16] $\implies$ compatibility

$$\mathrm{Clone}(g, d) := \{s \in [0,1]^g \,: \exists \text{ quantum channel } \Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g} \text{ s.t.}$$
$$\forall i \in [g], \quad \Phi_i(X) = s_i X + (1 - s_i)\frac{\operatorname{Tr} X}{d}\}$$

- Connection with tensor norms [BN22b]

Theorem ([BN18])

For all g and $d \geq 2^{\lceil (g-1)/2 \rceil}$, $\Gamma(g, d) = \Delta(g, d) = \mathrm{QC}_g$

Proof ideas

Inclusion of spectrahedra and (completely) positive maps

Theorem ([HKM13])

Let $A \in (\mathcal{M}_D^{sa}(\mathbb{C}))^g$, $B \in (\mathcal{M}_d^{sa}(\mathbb{C}))^g$ such that $\mathcal{D}_A(1)$ is bounded. Then, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ iff the unital linear map

$$\Phi : \mathrm{span}\{I, A_1, \ldots, A_g\} \to \mathcal{M}_d^{sa}(\mathbb{C})$$
$$A_i \mapsto B_i$$

is n-positive.

Sketch of the proof of the main theorem:

- Level 1: the extremal points of $\mathcal{D}_{\diamondsuit,g}(1)$ are $\pm e_i$
- The inclusion $\mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$ holds iff the unital map $\Phi : I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2 \mapsto 2E_i - I_d$ is CP
- Arveson's extension theorem [Pau02, Theorem 6.2]: Φ has a (completely) positive extension $\tilde{\Phi}$ to $\mathbb{R}^{2^g}$
- $C_\varepsilon := \tilde{\Phi}(\varepsilon)$ is a joint POVM for the E_i's, where $\{\varepsilon\}$ is a basis of $\mathbb{R}^{2^g}$

Maximally incompatible quantum effects

Lemma ([Hru16])

For $d = 2^k$, there exist $2k+1$ anti-commuting, self-adjoint, unitary matrices $F_1, \ldots, F_{2k+1} \in \mathcal{U}_d$. Moreover, 2^k is the smallest dimension where such a $(2k+1)$-tuple exists.

- For $k = 0$, take $F_1^{(0)} := [1]$
- For $k \geq 1$, define $F_i^{(k+1)} = \sigma_X \otimes F_i^{(k)}$ $\forall i \in [2k+1]$ and $F_{2k+2}^{(k+1)} = \sigma_Y \otimes I_{2^k}$, $F_{2k+3}^{(k+1)} = \sigma_Z \otimes I_{2^k}$
- These matrices satisfy, for all $x \in \mathbb{R}_+^g$, $\left\|\sum_{i=1}^g x_i F_i\right\|_\infty = \|x\|_2$, and $\left\|\sum_{i=1}^g x_i \bar{F}_i \otimes F_i\right\|_\infty = \|x\|_1$
- For d large enough, the maximally incompatible g-tuple of quantum effects in $\mathcal{M}_d$ is given by $E_i = (F_i + I_d)/2$

The take-home slide

Measurement compatibility in QM $\iff$ matrix diamond inclusion

- A measurement in QM (POVM): $A_i \in \mathcal{M}_d(\mathbb{C})$, $A_i \geq 0$, $\sum_i A_i = I_d$
- Measurements $(A_i), (B_j)$ are compatible if there exists a joint measurement (C_{ij}) having marginals A, B: $A_i = \sum_j C_{ij}$ and $B_j = \sum_i C_{ij}$
- Free spectrahedra: $\mathcal{D}_A := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^{g} X_i \otimes A_i \leq I_{nd}\}$
- Matrix diamond: $\mathcal{D}_{\diamondsuit,g} = \bigsqcup_{n=1}^{\infty} \{X \in \mathcal{M}_n^g : \sum_{i=1}^{g} \varepsilon_i X_i \leq I_n, \forall \varepsilon \in \{\pm 1\}^g\}$

Theorem

Let $E \in (\mathcal{M}_d^{sa})^g$ be g-tuple of selfadjoint matrices. Then:

$$E \text{ are quantum effects} \iff \mathcal{D}_{\diamondsuit,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$$

$$E \text{ are compatible quantum effects} \iff \mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$$

For all g, d, compatibility region $\Gamma(g,d) = \Delta(g,d)$ inclusion constants.

Steering inequality violation $\iff$ matrix cube inclusion

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