On spectral properties of random quantum channels

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My research

My research activities are focused on two main themes:

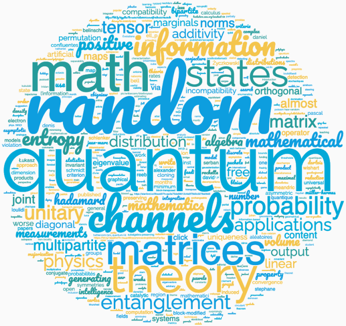
- quantum information theory
- random matrices, free probability

as well as on the interactions between these topics, more precisely the study of random quantum objects, such as quantum states and channels. Mainly, I tried to understand the properties of (random) matrices acting on vector spaces with a tensor product structure.

I have published 68 papers, mostly in mathematical physics, probability theory, and (multi)linear algebra journals

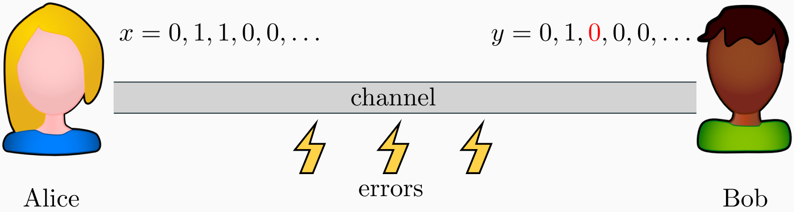
I have given 123 talks, among which 43 were invited talks at international workshops

I am supervising 1 PhD thesis and co-supervising 2 others



Capacity of channels

Classical channels



- Two parties, Alice and Bob want to communicate classically letters from the alphabet $\{1, 2, \ldots, d\}$
- Their communication channel is noisy:

$$\mathbb{P}[\text{Bob receives } j \mid \text{Alice sent } i] = M_{ij}$$

- Classical channels $\equiv$ Markov matrices acting on probability vectors
  - Positivity: for all i, j, $M_{ij} \geq 0$
  - Mass preservation: for all i, $\sum_j M_{ij} = 1$
- Example: bit flip channel $M = \begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix}$

Quantum channels

Channels	Deterministic	Noisy
Classical	$f : [d] \to [d]$	M Markov: $M_{ij} \geq 0$ and $\forall i$, $\sum_j M_{ij} = 1$
Quantum	$U \in \mathcal{U}(d)$	Φ completely positive, trace pres. map

- Classical channels (acting on probability vectors):
  - Positivity: for all i, j, $M_{ij} \geq 0$
  - Mass preservation: for all j, $\sum_i M_{ij} = 1$.
- Quantum channels: CPTP linear maps $\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$
  - CP - complete positivity: $\Phi \otimes \mathrm{id}_k$ is a positive map, $\forall k \geq 1$. Positivity:

$$X \text{ positive semi-definite} \implies \Psi(X) \text{ positive semi-definite}$$

  - TP - trace preservation: $\mathrm{Tr} \circ \Phi = \mathrm{Tr}$.
- Example: depolarizing channel $\Phi(X) = (1 - \varepsilon)X + \varepsilon(\mathrm{Tr}\, X/d)I_d$

Classical capacity of channels

- Classical capacity of a channel = the maximal rate at which classical information can be reliably transmitted through the channel

Theorem ([Sha48])

The classical capacity of a classical channel M is

$$C(M) = \max_X I(X;Y),$$

where I is the mutual information and $Y = M(X)$.

Theorem ([Hol73, SW97])

The classical capacity of a quantum channel Φ is

$$C(\Phi) = \lim_{r\to\infty} \frac{\chi(\Phi^{\otimes r})}{r},$$

where χ is the Holevo capacity of a channel:

$$\chi(\Psi) = \max_{\{p_i,\rho_i\}} H\left(\sum_i p_i \Phi(\rho_i)\right) - \sum_i p_i H(\Phi(\rho_i))$$

Importance of additivity

- p-Minimal Output Entropy of a quantum channel

$$\begin{aligned} H^p_{\min}(\Phi) &= \min_{\rho \in \mathcal{M}^{1,+}_{\text{in}}(\mathbb{C})} H^p(\Phi(\rho)) \\ &= \min_{x \in \mathbb{C}^{\text{in}}} H^p(\Phi(P_x)) \end{aligned}$$

- Is the p-MOE additive ?

$$H^p_{\min}(\Phi \otimes \Psi) = H^p_{\min}(\Phi) + H^p_{\min}(\Psi) \quad \forall \Phi, \Psi$$

- Simple formula for the (classical) capacity of quantum channels: if additivity holds, then there is no need to use inputs entangled over multiple uses of Φ.
- Equivalence of additivity questions [Sho04]
  1. additivity of the Holevo capacity χ
  2. additivity of the minimum output entropy (MOE)
  3. (strong super-) additivity of the entanglement of formation E_F.
- MOE is NOT additive: first shown for $p > 4.79$ [WH02], then for $p > 1$ [HW08], and finally for $p = 1$ [Has09]
- Difficult, mathematically challenging problem

Random quantum channels

Structure of quantum channels

Theorem [Stinespring-Kraus-Choi]

Let $\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ be a linear map. TFAE:

1. The map Φ is completely positive and trace preserving (CPTP).
2. [Stinespring] There exist an integer s ($s = d_1 d_2$ suffices) and an isometry $W : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2} \otimes \mathbb{C}^s$ such that
$$\Phi(X) = [\mathrm{id}_{d_2} \otimes \mathrm{Tr}_s](WXW^*).$$
3. [Kraus] There exist operators $A_1, \ldots, A_s \in \mathcal{M}_{d_2 \times d_1}$ satisfying $\sum_i A_i^* A_i = I_{d_1}$ such that
$$\Phi(X) = \sum_{i=1}^{s} A_i X A_i^*.$$
4. [Choi] The Choi matrix C_Φ is positive semidefinite, where
$$C_\Phi := \sum_{i,j=1}^{d_1} E_{ij} \otimes \Phi(E_{ij}) \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2}$$
and $[\mathrm{id}_{d_1} \otimes \mathrm{Tr}_{d_2}](C_\Phi) = I_{d_1}$.

Random quantum channels

There exist several natural candidates for probability distributions on the set of quantum channels $\{\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}\}$

1. The Lebesgue measure: the set of quantum channels is convex and compact, having real dimension $d_1^2 d_2^2 - d_1^2$. Normalize the volume measure to obtain a probability distribution $\mu_{d_1,d_2}^{Lebesgue}$
2. Pick the isometry W in the Stinespring decomposition at random: W is a Haar-random isometry $\mathbb{C}^{d_1} \to \mathbb{C}^{d_2} \otimes \mathbb{C}^s$. We obtain a probability distribution $\mu_{d_1,d_2;s}^{Stinespring}$, where $s \geq 1$ is an integer such that $d_1 \leq sd_2$
3. Pick the Kraus operators A_i at random: G_i are i.i.d. $d_2 \times d_1$ Ginibre matrices, define $A_i = G_i S^{-1/2}$, with $S = \sum_{i=1}^s G_i^* G_i$. We obtain a probability distribution $\mu_{d_1,d_2;s}^{Kraus}$, where $s \geq 1$ is an integer such that $d_1 \leq sd_2$
4. Pick the Choi matrix at random: $\tilde{C}$ is a Wishart matrix of parameters $d_1 d_2, s)$, define $C := [I \otimes T^{-1/2}]\tilde{C}[I \otimes T^{-1/2}]^*$, with $T = [\mathrm{Tr} \otimes \mathrm{id}]\tilde{C}$. We obtain a probability distribution $\mu_{d_1,d_2;s}^{Choi}$, where $s \geq 1$ is any real number $s \geq d_1 d_2$, or an integer $s \geq d_1/d_2$

Equivalence of probability measures

Theorem ([KNP+21])

The above distributions are identical, when the respective parameters match:

$$\mu_{d_1,d_2}^{Lebesgue} \in \left\{ \mu_{d_1,d_2;s}^{Stinespring} \right\}_{\substack{s \in \mathbb{N} \\ s \geq d_2/d_1}} = \left\{ \mu_{d_1,d_2;s}^{Kraus} \right\}_{\substack{s \in \mathbb{N} \\ s \geq d_2/d_1}} \subset \left\{ \mu_{d_1,d_2;s}^{Choi} \right\}_{s \in \mathcal{S}_{d_1,d_2}}$$

where

$$\mathcal{S}_{d_1,d_2} := \left\{ \left\lceil \frac{d_1}{d_2} \right\rceil, \left\lceil \frac{d_1}{d_2} \right\rceil + 1, \ldots, d_1 d_2 - 1 \right\} \sqcup [d_1 d_2, +\infty)$$

The Lebesgue measure is obtained for $s = d_1 d_2$.

Computationally, the random Kraus operators procedure is the cheapest; mathematically, the random isometry procedure is the more interesting and easier to deal with, since no normalization procedure is needed, and the structure of Haar random isometries is well understood

Model of interest

Here, we focus on random quantum channels coming from random isometries, with the following parameters.

- in $= tnk$,
- out $= k$,
- anc $= n$,

where $n, k \in \mathbb{N}$ and $t \in (0,1)$. In general, we shall assume that

- $n \to \infty$
- k is fixed
- t is fixed.

In other words, we are interested in $\Phi : \mathcal{M}_{tnk}(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$,

$$\Phi(\rho) = [\mathrm{id}_k \otimes \mathrm{Tr}_n](V\rho V^*),$$

where V is a random isometry obtained by keeping the first tnk columns of a $nk \times nk$ Haar random unitary.

How to get counterexamples ?

- Choose Φ to be random and $\Psi = \bar{\Phi}$; this way, $H^p_{\min}(\Psi) = H^p_{\min}(\Phi)$.
- Bound

$$H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi).$$

MOE of a single channel

Strategy for B_1

- Remember: we want

$$H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi)$$

- We shall do more: we compute the exact limit (as $n \to \infty$) of $H^p_{\min}(\Phi)$

Theorem ([BCN12, BCN16])

For all $p \geq 1$, almost surely

$$\lim_{n\to\infty} H_p^{min}(\Phi) = H_p(a, b, b, \ldots, b),$$

where a, b do not depend on p, $b = (1-a)/(k-1)$ and $a = \varphi(1/k, t)$ with

$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \text{if } s+t<1; \\ 1 & \text{if } s+t \geq 1 \end{cases}$$

- Proof strategy: prove that the set of *all* eigenvalue vectors of outputs converges to a deterministic limit, described by a norm defined using free compression

Bounds for the tensor product

Strategy for B_2

- Remember: we want

$$H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi).$$

- Use trivial bound $H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq H^p\left([\Phi \otimes \bar{\Phi}](X_{12})\right)$, for a particular choice of $X_{12} \in \mathcal{M}_{tnk}(\mathbb{C}) \otimes \mathcal{M}_{tnk}(\mathbb{C})$.
- $X_{12} = X_1 \otimes X_2$ do not yield counterexamples $\Rightarrow$ choose a maximally entangled state

$$X_{12} = E_{tnk} = \left(\frac{1}{\sqrt{tnk}} \sum_{i=1}^{tnk} e_i \otimes e_i\right) \left(\frac{1}{\sqrt{tnk}} \sum_{j=1}^{tnk} e_j \otimes e_j\right)^* .$$

- Bound entropies of the (random) density matrix

$$Z_n = [\Phi \otimes \overline{\Phi}](E_{tnk}) \in \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_k(\mathbb{C}).$$

Main result - output eigenvalues

Theorem ([CN10])

For all k, t, almost surely as $n \to \infty$, the eigenvalues of $Z_n = [\Phi \otimes \bar{\Phi}](E_{tnk})$ converge to

$$\left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \text{ times}} \right) \in \Delta_{k^2}.$$

- Previously known bound (deterministic, comes from linear algebra): for all t, n, k, the largest eigenvalue of Z_n is at least t.
- Two improvements:
  1. "better" largest eigenvalue,
  2. knowledge of the whole spectrum.
- Precise knowledge of eigenvalues $\rightsquigarrow$ optimal estimates for entropies.
- However, smaller eigenvalues are the "worst possible".

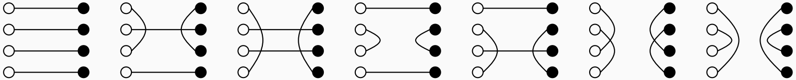
Proof strategy for a.s. spectrum Z_n

- Use the method of moments
  1. Convergence in moments:
  $$\mathbb{E}\mathrm{Tr}(Z_n^p) \to \left(t + \frac{1-t}{k^2}\right)^p + (k^2-1)\left(\frac{1-t}{k^2}\right)^p;$$
  2. Borel-Cantelli for a.s. convergence:
  $$\sum_{n=1}^{\infty} \mathbb{E}\left[\left(\mathrm{Tr}(Z_n^p) - \mathbb{E}\mathrm{Tr}(Z_n^p)\right)^2\right] < \infty.$$
- We need to compute moments $\mathbb{E}\left[\mathrm{Tr}(Z_n^{p_1})^{q_1} \cdots \mathrm{Tr}(Z_n^{p_s})^{q_s}\right]$.
- Use the Weingarten formula to compute the unitary averages.

More than two channels

Multiple conjugate channels

- Consider a tensor product of $2r$ channels: $\Psi = \Phi^{\otimes r} \otimes \bar{\Phi}^{\otimes r}$
- A partial permutation β is a bijection form a subset of $[r]$ to another subset of $[r]$. To such a β, we associate an operator which pairs factors:



- A sequence of vectors $\psi_n \in (\mathbb{C}^n)^{\otimes 2r}$ is called well behaved if for all partial permutations $\beta \in \hat{\mathcal{S}}_r$, one has

$$\lim_{n\to\infty} \langle \psi_n | \tilde{T}_\beta^{(n)} | \psi_n \rangle = a_\beta \in [0,1]$$

Theorem ([FN14])

Among all sequences of well-behaved input states, the ones having a minimal output entropy are (asymptotically) a tensor product of maximally entangled states (where the matching $\Phi \leftrightarrow \bar{\Phi}$ of the conjugate channels is given by an arbitrary full permutation $\pi \in \mathcal{S}_r$).

- It seems that multipartite entanglement does not help with additivity violations

Multiplicative bounds

- For a channel Φ and an entropy parameter p, define the *p*-additivity rate of Φ

$$\alpha_p(\Phi) := \sup\left\{a \in [0,1] \,:\, \liminf_{r\to\infty} \frac{1}{r} H_p^{\min}(\Phi^{\otimes r}) \geq a H_p^{\min}(\Phi)\right\}$$

- Consider the following quantities associated to a quantum channel Φ:

$$B_C(\Phi) = \|C_\Phi\| \qquad B_{C\Gamma}(\Phi) = \|C_\Phi^\Gamma\| \qquad B_{C^c\Gamma}(\Phi) = \|C_{\Phi^c}^\Gamma\|$$

  where Φ^c is the complementary channel of Φ and C_Φ is the Choi matrix of Φ
- These are multiplicative quantities: $B_*(\Phi_1 \otimes \Phi_2) = B_*(\Phi_1)B_*(\Phi_2)$
- They are lower bounds for the 2-Rényi minimum output entropy:

$$H_2^{\min}(\Phi) \leq -\log B_*(\Phi) \implies \alpha_p(\Phi) \geq \frac{-\log B_*}{H_p^{\min}(\Phi)} \qquad \forall p \in [0,2]$$

- A similar conclusion holds for the quantities

$$B_{M\Gamma}(\Phi) = \|M_\Phi^\Gamma\| \qquad B_I(\Phi) = \|\Phi(I)\|$$

  and for the $(p=\infty)$-additivity rate of Φ. Above $M = VV^*$ is the projection on the range of the Strinespring isometry V defining Φ

Additivity violations

Recall

$$H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi)$$

Theorem ([CN10])

For all k, t, almost surely as $n \to \infty$, if $Z_n = (\Phi \otimes \bar{\Phi})(E_{tnk})$

$$\mathrm{spec}(Z_n) \to \left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \ \textit{times}} \right) \in \Delta_{k^2}.$$

Theorem ([BCN12, BCN16])

For all $p \geq 1$,

$$\lim_{n\to\infty} H_p^{min}(\Phi) = H_p(a, b, b, \ldots, b),$$

where $b = (1-a)/(k-1)$ and $a = \varphi(1/k, t)$ with

$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \textit{if } s + t < 1; \\ 1 & \textit{if } s + t \geq 1. \end{cases}$$

Conclusion and further directions

Theorem ([BCN16])

Using the limit for $H^{\min}(\Phi)$ and the upper bound for $H^{\min}(\Phi)$, the lowest dimension for which a violation of the additivity can be observed is $k = 183$. For large k, violations of size $1 - \varepsilon$ bits can be obtained.

Going further:

1. Larger violations, smaller k? Finite n done in [CP22]
2. Other asymptotic regimes: $k \sim n$ (i.e. same input and output space)
3. Use $\Psi \neq \bar{\Phi}$
4. For $\Phi \otimes \bar{\Phi}$, compute the actual limit of $H^{\min}(\Phi \otimes \bar{\Phi})$, and not just an upper bound using the maximally entangled state E_d
5. Regularization: $\Phi^{\otimes r}$. What is a "good" input state?
6. Additivity for symmetric channels? YES for diagonal unitary covariant channels, $p \geq 2$ (work in progress with Sang-Jun Park)

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