Matrix convex sets with polytope base, quantum Latin squares, and compatibility

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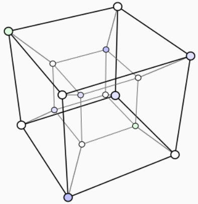
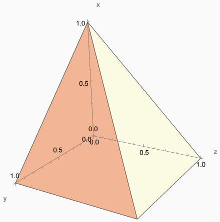
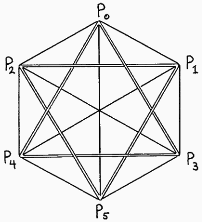
Outline







We show that some fundamental problems in quantum information theory are related to matrix convex sets built on polytopes. The amount by which the maximal matrix convex set has to be shrunk to fit inside the minimal one has an operational meaning.

Matrix convex sets

Matrix convex sets

We consider free sets:

$$\mathcal{F} = \bigsqcup_{n \geq 1} \mathcal{F}_n,$$

where $\mathcal{F}_n \subseteq (\mathcal{M}_n^{\mathrm{sa}})^g$.

The free set $\mathcal{F}$ is matrix convex [DDOSS17] if it is closed under direct sums and unital completely positive maps:

- $(A_1, \ldots, A_g) \in \mathcal{F}_m$, $(B_1, \ldots, B_g) \in \mathcal{F}_n \implies (A_1 \oplus B_1, \ldots, A_g \oplus B_g) \in \mathcal{F}_{m+n}$.
- $(A_1, \ldots, A_g) \in \mathcal{F}_m$, $\Phi : \mathcal{M}_m \to \mathcal{M}_n$ UCP $\implies (\Phi(A_1), \ldots, \Phi(A_g)) \in \mathcal{F}_n$

UCP maps $\Phi : \mathcal{M}_m \to \mathcal{M}_n$ are maps such that $\Phi \otimes \mathrm{id}_k$ is positive for all $k \geq 1$ and $\Phi(I_m) = I_n$.

Alternatively, $\Phi(X) = \sum_i K_i^* X K_i$ such that $\sum_i K_i^* K_i = I_n$, $K_i \in \mathcal{M}_{m,n}$, see [Wat18].

Minimal and maximal matrix convex sets

- Unless $\mathcal{C}$ is a simplex, there are arbitrarily many different matrix convex sets with the same $\mathcal{F}_1 = \mathcal{C}$. However, there is a largest and a smallest such set:
- For a closed convex set $\mathcal{C}$,

$$\mathcal{W}_n^{\max}(\mathcal{C}) := \left\{ X \in (\mathcal{M}_n^{\mathrm{sa}})^g : \sum_{i=1}^{g} c_i X_i \leq \alpha I \ \forall (\alpha, c) \text{ supp. hyperplanes for } \mathcal{C} \right\}$$

- For a closed convex set $\mathcal{C}$,

$$\mathcal{W}_n^{\min}(\mathcal{C}) := \left\{ \sum_j X = z_j \otimes Q_j \in (\mathcal{M}_n^{\mathrm{sa}})^g : z_j \in \mathcal{C},\ Q_j \geq 0 \ \forall j, \sum_j Q_j = I_n \right\}$$

- Observe $\mathcal{W}_1^{\max}(\mathcal{C}) = \mathcal{C} = \mathcal{W}_1^{\min}(\mathcal{C})$. $\mathcal{W}^{\max}(\mathcal{C})$ quantizes hyperplanes, $\mathcal{W}^{\min}(\mathcal{C})$ quantizes extreme points [FNT17].

Definition

Let $d, g \in \mathbb{N}$ and $\mathcal{C} \subset \mathbb{R}^g$ closed convex. The inclusion set of $\mathcal{C}$ is defined as

$$\Delta_{\mathcal{C}}(d) := \left\{ s \in [0,1]^g : s \cdot \mathcal{W}_d^{\max}(\mathcal{C}) \subseteq \mathcal{W}_d^{\min}(\mathcal{C}) \right\}.$$

Measurement compatibility

Quantum states and measurements

- Motivation: Classical state $\rightsquigarrow$ probability distributions: $p \in \mathbb{R}^d$, $p \geq 0$, $\sum_i p_i = 1$.
- Quantum states $\rightsquigarrow$ density matrices: $\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$.
- A measurement apparatus is a device which takes in a quantum state and yields a classical measurement result, i.e. a label $i \in \{1, \ldots, k\}$.
- Outcome probabilities are given by the Born rule.
- Measurements: Tuples of matrices $(E_1, \ldots, E_k)$ such that $(\operatorname{Tr}[\rho E_1], \ldots, \operatorname{Tr}[\rho E_k])$ is a probability distribution for all states ρ.
  - $\operatorname{Tr}[\rho E_i] \in \mathbb{R} \rightsquigarrow E_i = E_i^*$.
  - $\operatorname{Tr}[\rho E_i] \geq 0 \rightsquigarrow E_i \geq 0$.
  - $\sum_i \operatorname{Tr}[\rho E_i] = 1 \rightsquigarrow \sum_i E_i = I_d$.
- Tuples of PSD matrices summing to identity are called positive operator-valued measures (POVMs).
- Examples:
  - basis measurements $(|e_i\rangle\langle e_i|)_{i \in [d]}$;
  - trivial measurements $(p_j I_d)_{j \in [k]}$ for a probability vector p.

Quantum measurements: Compatibility

- Quantum measurements $\rightsquigarrow$ give the probabilities of the classical outcomes when a quantum state enters a measurement apparatus. Mathematically, measurements are modeled by POVMs.

Definition

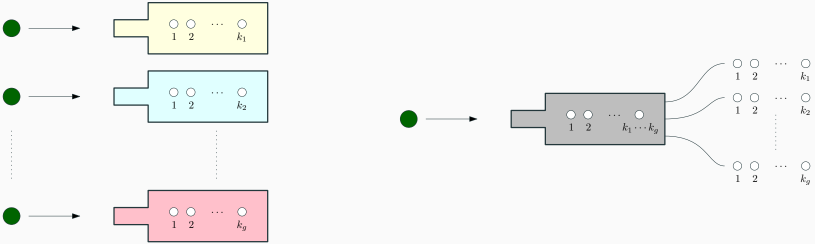
Two POVMs, $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_l)$, are called compatible if there exists a third POVM $C = (C_{ij})_{i \in [k], j \in [l]}$ such that

$$\forall i \in [k], \quad A_i = \sum_{j=1}^{l} C_{ij} \qquad \text{and} \qquad \forall j \in [l], \quad B_j = \sum_{i=1}^{k} C_{ij}.$$

The definition generalizes to g-tuples of POVMs $A^{(1)}, \ldots, A^{(g)}$, having respectively $k_1, \ldots k_g$ outcomes, where the joint POVM C has outcome set $[k_1] \times \cdots \times [k_g]$.

- Other way to say that: jointly measurable [HMZ16].

What does it mean?



- Compatible measurements can be simulated by a single joint measurement, by classically post-processing its outputs $A_i^{(j)} = \sum_\lambda p_j(i|\lambda) C_\lambda$.
- Examples:
  1. Trivial POVMs $A = (p_i I_d)$ and $B = (q_j I_d)$ are compatible.
  2. Commuting POVMs $[A_i, B_j] = 0$ are compatible.
  3. If the POVM A is projective, then A and B are compatible if and only if they commute.

Noisy POVMs

- POVMs can be made compatible by adding noise, i.e. mixing in trivial POVMs.
- Example: dichotomic POVMs and white noise, $s \in [0,1]$:

$$(E, I-E) \mapsto s(E, I-E) + (1-s)(\frac{I}{2}, \frac{I}{2}) \qquad \text{or} \qquad E \mapsto sE + (1-s)\frac{I}{2}.$$

- Taking $s = 1/2$ suffices to render any pair of dichotomic POVMs compatible $\rightsquigarrow$ define $C_{ij} := (E_i + F_j)/4$.
- From now on, we focus on dichotomic (YES/NO) POVMs.

Definition ([BN18])

The compatibility region for g measurements on $\mathbb{C}^d$ is the set

$$\Gamma(g,d) := \{s \in [0,1]^g \ : \ \text{for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}), \\ \text{the noisy versions } s_i E_i + (1-s_i) I_d/2 \text{ are compatible}\}$$

Compatibility region

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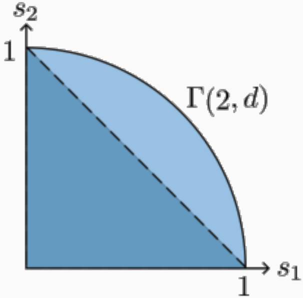
- The set $\Gamma(g,d)$ is convex.
- For all $i \in [g]$, $e_i \in \Gamma(g,d)$: every measurement is compatible with $g-1$ trivial measurements.
- For $d \geq 2$, $(1,1,\ldots,1) \notin \Gamma(g,d)$: there exist incompatible measurements.
- For all $d \geq 2$, $\Gamma(2,d)$ is a quarter-circle.



Generally speaking, the set $\Gamma(g,d)$ tells us how robust (to noise) is the incompatibility of g dichotomic measurements on $\mathbb{C}^d$.

Measurement compatibility and matrix convex sets

Measurement compatibility revisited

From now on, we concentrate on measurements with two outcomes and identify $E^{(i)} = \{E_i, I - E_i\}$ with E_i.

Theorem ([BN18])

Let $E_1, \ldots, E_g$ be arbitrary self-adjoint operators. Define

$$A = \sum_{j=1}^{g} e_j \otimes (2E_j - I).$$

Then, $A \in \mathcal{W}_d^{\max}(\mathcal{B}(\ell_\infty^g))$ if and only if $\{E_j\}_{j\in[g]}$ is a collection of POVMs. Moreover, $A \in \mathcal{W}_d^{\min}(\mathcal{B}(\ell_\infty^g))$ if and only if $\{E_j\}_{j\in[g]}$ is a collection of compatible POVMs.

Proof sketch

- $\mathcal{W}_d^{\mathrm{max}}(\mathcal{B}(\ell_\infty^g))$ is given in terms of hyperplanes. Have to verify $-I \leq A_i = 2E_i - I \leq I \implies 0 \leq E_i \leq I$.
- Reminder:

$$\mathcal{W}_n^{\mathrm{min}}(\mathcal{B}(\ell_\infty^g)) := \Big\{ X = \sum_j z_j \otimes Q_j \in (\mathcal{M}_n^{\mathrm{sa}})^g : z_j \in \mathcal{C}\ \forall j,\ Q \text{ POVM} \Big\}.$$

- Going to extreme points:

$$2E_j - I = \sum_{\varepsilon \in \{\pm 1\}} \varepsilon(j) Q_\varepsilon.$$

- Using $\sum_\varepsilon Q_\varepsilon = I$:

$$E_j = \sum_{\varepsilon \in \{\pm 1\} : \varepsilon(j) = 1} Q_\varepsilon.$$

- $\{Q_\varepsilon\}_\varepsilon$ is a joint POVM.

Inclusion sets and compatibility regions

Let $\Delta_\square$ be the inclusion set of the hypercube:

$$\Delta_\square(g,d) := \left\{ s \in [0,1]^g : s \cdot \mathcal{W}_d^{\max}(\mathcal{B}(\ell_\infty^g)) \subseteq \mathcal{W}_d^{\min}(\mathcal{B}(\ell_\infty^g)) \right\}.$$

Theorem ([BN18])

Let g, $d \in \mathbb{N}$. Let $s \in [0,1]^g$. Then, $\{s_i E_i + (1-s_i)I/2\}_{i\in[g]}$ is a collection of compatible POVMs for all POVMs $\{E_i\}_{i\in[g]}$, if and only if $s \in \Delta_\square(g,d)$. An equivalent way to phrase this is $\Gamma(g,d) = \Delta_\square(g,d)$.

- This follows from the computation

$$A_i' = 2(s_i E_i + (1-s_i)I/2) - I = s_i(2E_i - I) = s_i A_i.$$

- So adding noise means scaling the tensor A and hence $s \cdot \mathcal{W}_d^{\max}(\mathcal{B}(\ell_\infty^g))$ is the set of noisy measurements.
- Thus, $s \cdot A \in \mathcal{W}_d^{\min}(\mathcal{B}(\ell_\infty^g))$ means the noisy measurements are compatible.

Polytope compatibility

work in progress, “soon” on the arXive

Polytope compatibility

Definition

Let $\mathcal{P}$ be a polytope in $\mathbb{R}^g$ such that $0 \in \operatorname{int} \mathcal{P}$. Let

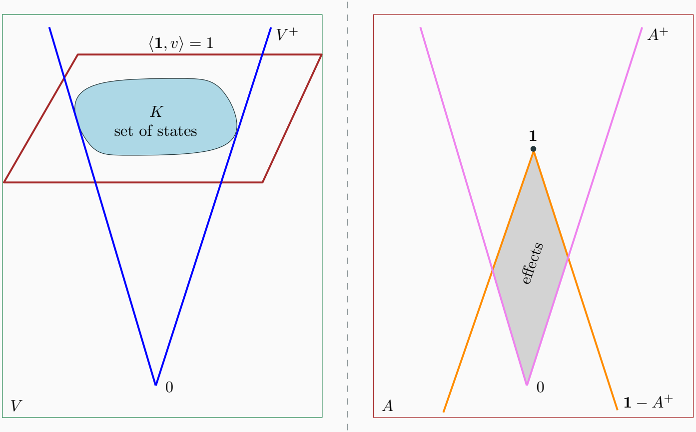
$$A = (A_1, \dots, A_g) \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^g \cong \mathbb{R}^g \otimes \mathcal{M}_d^{sa}(\mathbb{C})$$

a g-tuple of Hermitian matrices. Then, A are *$\mathcal{P}$-operators* if and only if $A \in \mathcal{W}_d^{\max}(\mathcal{P})$. Moreover, A are *$\mathcal{P}$-compatible* if and only if $A \in \mathcal{W}_d^{\min}(\mathcal{P})$.

Motivation:

- A are $\mathcal{B}(\ell_\infty^g)$-operators if and only if $\frac{1}{2}(A_i + I)$ are dichotomic POVMs.
- A are $\mathcal{B}(\ell_\infty^g)$-compatible if and only if $\frac{1}{2}(A_i + I)$ are compatible dichotomic POVMs.

Interlude: General Probabilistic Theories



- A GPT [Lam18] is a triple $(V, V^+, \mathbb{1})$, where V is a vector space, $V^+ \subseteq V$ is a cone, and $\mathbb{1}$ is a linear form on V; $A = V^*$, $A^+ = (V^+)^*$, and $\mathbb{1} \in A^+$
- The set of states $K := V^+ \cap \mathbb{1}^{-1}(\{1\})$

Equivalent formulation

Theorem

Let $d, g, k \in \mathbb{N}$ and let $\mathcal{P}$ be a polytope with k extremal points $v_1, \ldots, v_k \in \mathbb{R}^g$ such that $0 \in \operatorname{int} \mathcal{P}$. Let $A = (A_1, \ldots, A_g) \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^g$ be a g-tuple of Hermitian matrices. Let us consider the map $\mathcal{A} : \mathcal{M}_d^{\mathrm{sa}} \to \mathbb{R}^g$,

$$\mathcal{A}(X) = (\mathrm{Tr}[A_1 X], \ldots, \mathrm{Tr}[A_g X]).$$

Then,

1. *A are $\mathcal{P}$-operators if and only if $\mathcal{A}$ is a channel between $(\mathcal{M}_d^{\mathrm{sa}}, \mathrm{PSD}_d, \mathrm{Tr})$ and $(V(\mathcal{P}), V(\mathcal{P})^+, \mathbb{1}_{\mathcal{P}})$.*
2. *A are $\mathcal{P}$-compatible if and only if in addition $\mathcal{A}$ factors through the k-simplex Δ_k.*

Interpretation: $\mathcal{P}$ defines some kind of allowed post-processing.
In the case of $\mathcal{B}(\ell_\infty^g)$: Classical post-processing.

Magic squares

A magic square is a collection of positive operators A_{ij}, $i, j \in [N]$, such that

$$\begin{array}{ccccccccc}
A_{11} & + & A_{12} & + & \dots & + & A_{1N} & = I \\
+ & & + & & & & + & \\
\vdots & & \vdots & & \ddots & & \vdots & \vdots \\
+ & & + & & & & + & \\
A_{N1} & + & A_{N2} & + & \dots & + & A_{NN} & = I \\
\| & & \| & & & & \| & \\
I & & I & & \dots & & I &
\end{array}$$

The magic square is said to be semiclassical [DICDN20] if

$$A = \sum_{i,j \in [N]} E_{ij} \otimes A_{ij} = \sum_{\pi \in \mathcal{S}_N} P_\pi \otimes Q_\pi,$$

where P_π is the permutation matrix associated to π and $\{Q_\pi\}_\pi$ is a POVM.

Birkhoff polytope compatibility

Definition

For a given $N \geq 2$, the Birkhoff body $\mathcal{B}_N$ is defined as the set of $(N-1) \times (N-1)$ truncations of $N \times N$ bistochastic matrices, shifted by J/N:

$$\mathcal{B}_N = \{A^{(N-1)} - J_{N-1}/N \,:\, A \in \mathcal{M}_N(\mathbb{R}) \text{ bistochastic}\} \subset \mathbb{R}^{(N-1)^2}.$$

To a $(N-1)^2$-tuple of selfadjoint matrices A, we associate the $N \times N$ block-bistochastic matrix $\tilde{A} \in \mathcal{M}_N(\mathcal{M}_d(\mathbb{C}))$ obtained by filling in the last row and column of the matrix $I/N + A_{ij}$ to have sums equal to I_d.

Theorem

Consider a $(N-1)^2$-tuple of selfadjoint matrices $A \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^{(N-1)^2}$ and the corresponding matrix $\tilde{A} \in \mathcal{M}_N(\mathcal{M}_d(\mathbb{C}))$. Then, the matrix $\tilde{A}$ is a magic square if and only if $A - I/N$ are $\mathcal{B}_N$-operators, i.e. $A - I/N \in \mathcal{W}_d^{\max}(\mathcal{B}_N)$. Moreover, the matrix $\tilde{A}$ is a semiclassical magic square if and only if $A - I/N$ are $\mathcal{B}_N$-compatible, i.e. $A - I/N \in \mathcal{W}_d^{\min}(\mathcal{B}_N)$.

Relation to measurement incompatibility

Is being a semiclassical magic square the same as being compatible? No.

$\frac{1}{2}\lvert 0\rangle\langle 0\rvert$	$\frac{1}{2}\lvert 1\rangle\langle 1\rvert$	0	$\frac{1}{2}I_2$
$\frac{1}{2}\lvert 1\rangle\langle 1\rvert$	$\frac{1}{2}\lvert 0\rangle\langle 0\rvert$	$\frac{1}{2}I_2$	0
0	$\frac{1}{2}I_2$	$\frac{1}{2}\lvert +\rangle\langle +\rvert$	$\frac{1}{2}\lvert -\rangle\langle -\rvert$
$\frac{1}{2}I_2$	0	$\frac{1}{2}\lvert -\rangle\langle -\rvert$	$\frac{1}{2}\lvert +\rangle\langle +\rvert$

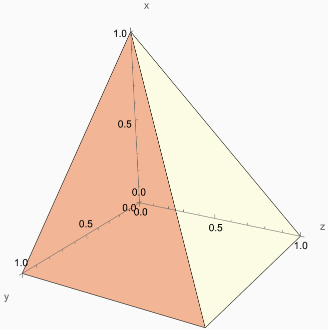
These measurements are compatible, but they do not form a semiclassical magic square.

Reason: $\mathcal{B}_N$-compatibility restricts the post-processing to $p_i(j|\lambda) = p_j(i|\lambda)$. This enforces special structure [GB19] in the joint POVM.

Measurement compatibility with shared effects

Can we generalize the magic square example?

$$\mathcal{P} = (-1/3, -1/3, -1/3) + \mathrm{conv}\{((1,0,0),(0,0,0),(0,0,1),(0,1,0),(0,1,1)\}.$$



Consider $(A, B, C) \in (\mathcal{M}_d(\mathbb{C})^{sa})^3$. Then, we have $(A, B, C) + 1/3(I, I, I) \in \mathcal{W}_d^{\max}(\mathcal{P})$ if and only if both $(A, B, I_d - A - B)$ and $(A, C, I_d - A - C)$ are POVMs.

Measurement compatibility with shared effects, continued

When does $(A, B, C) + 1/3(I, I, I) \in \mathcal{W}_d^{\min}(\mathcal{P})$ hold? Equivalent to the existence of a joint measurement such that

$$\begin{array}{|c|c|c|l}
\hline
Q_1 & 0 & 0 & = A \\
\hline
0 & Q_5 & Q_4 & = B \\
\hline
0 & Q_3 & Q_2 & = I_d - A - B \\
\hline
\multicolumn{1}{c}{= A} & \multicolumn{1}{c}{= C} & \multicolumn{1}{c}{= I_d - A - C} &
\end{array}$$

Not all compatible POVMs with common first effect are $\mathcal{P}$-compatible, check

$$\left(\frac{1}{2}I_2, \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|\right) \qquad \text{and} \qquad \left(\frac{1}{2}I_2, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -|\right).$$

Indeed, the unique joint measurement for the POVMs above is

0	$\frac{1}{2}\lvert+\rangle\langle+\rvert$	$\frac{1}{2}\lvert-\rangle\langle-\rvert$
$\frac{1}{2}\lvert 0\rangle\langle 0\rvert$	0	0
$\frac{1}{2}\lvert 1\rangle\langle 1\rvert$	0	0

Summary

- Measurement incompatibility can be phrased as inclusion of matrix convex sets. Base set: cube.
- Noise robustness corresponds to inclusion constants.
- Generalization: $\mathcal{P}$-operators and $\mathcal{P}$-compatible operators.
- Examples include magic squares and compatibility with shared elements (under restricted post-processing).

Can we find more tasks in quantum information theory which can be formulated as $\mathcal{P}$-compatibility?

Compute inclusion constants/sets for general polytopes. These have operational interpretation for the (restricted) compatibility of POVMs with shared elements.

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