Diagonal Unitary and Orthogonal Symmetries in Quantum Theory

joint work with Satvik Singh (Cambridge UK)

A graphical calculus for integration over random diagonal unitary matrices - LAA 613 (2021) [arXiv:2010.07898]
Diagonal unitary and orthogonal symmetries in quantum theory - Quantum 5, 519 (2021) [arXiv:2112.11123]
The PPT² conjecture holds for all Choi-type maps - Annales Henri Poincaré 23 (2022) [arxiv:2011.03809]

Ion Nechita (CNRS, LPT Toulouse)

LABORATOIRE DE PHYSIQUE THÉORIQUE TOULOUSE

FERMI

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Plan of the talk

1. Unitary symmetry in quantum information

2. Diagonal unitary / orthogonal symmetry

3. Positive semidefinite bipartite operators

Unitary symmetry in quantum information

Symmetry in quantum theory

- Quantum states are modeled mathematically by density matrices

$$\{\rho \in \mathscr{M}_d(\mathbb{C}) \;:\; \rho \geq 0 \text{ and } \operatorname{Tr} \rho = 1\}$$

- Unitary operators encode time evolution

$$\rho \mapsto U\rho U^*, \quad \text{for } U \in \mathscr{U}_d$$

- We would like to consider quantum states which are left invariant by a certain class of unitary operators

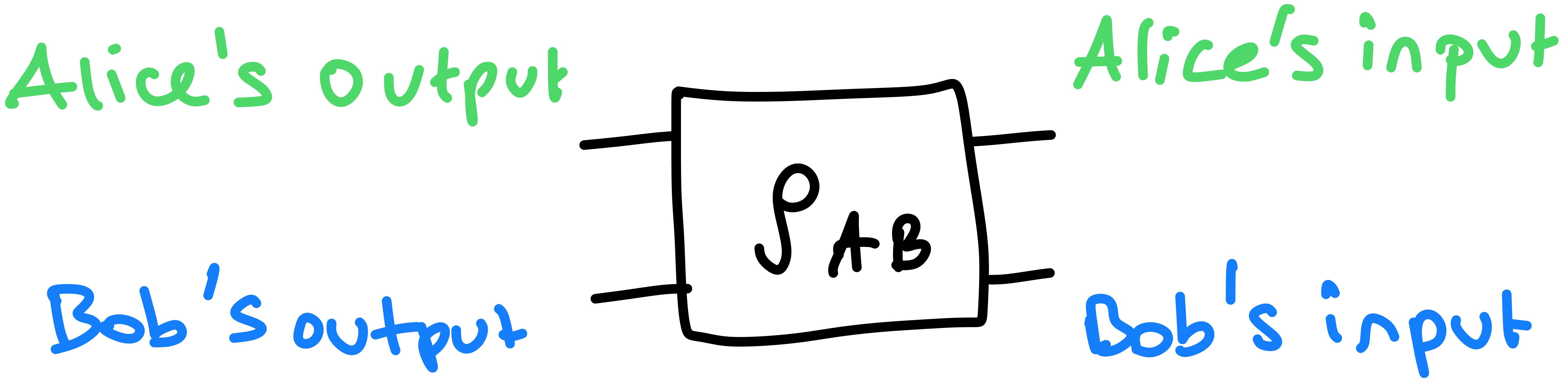
Example. If $UXU^* = X$ for all $U \in \mathscr{U}_d$, then X must be a scalar matrix, i.e. $X = cI_d$ for some constant $c \in \mathbb{C}$. For density matrices, $c = 1/d$.

Bipartite operators

- Quantum entanglement is of the key features of quantum theory. It appears when one considers two quantum systems.

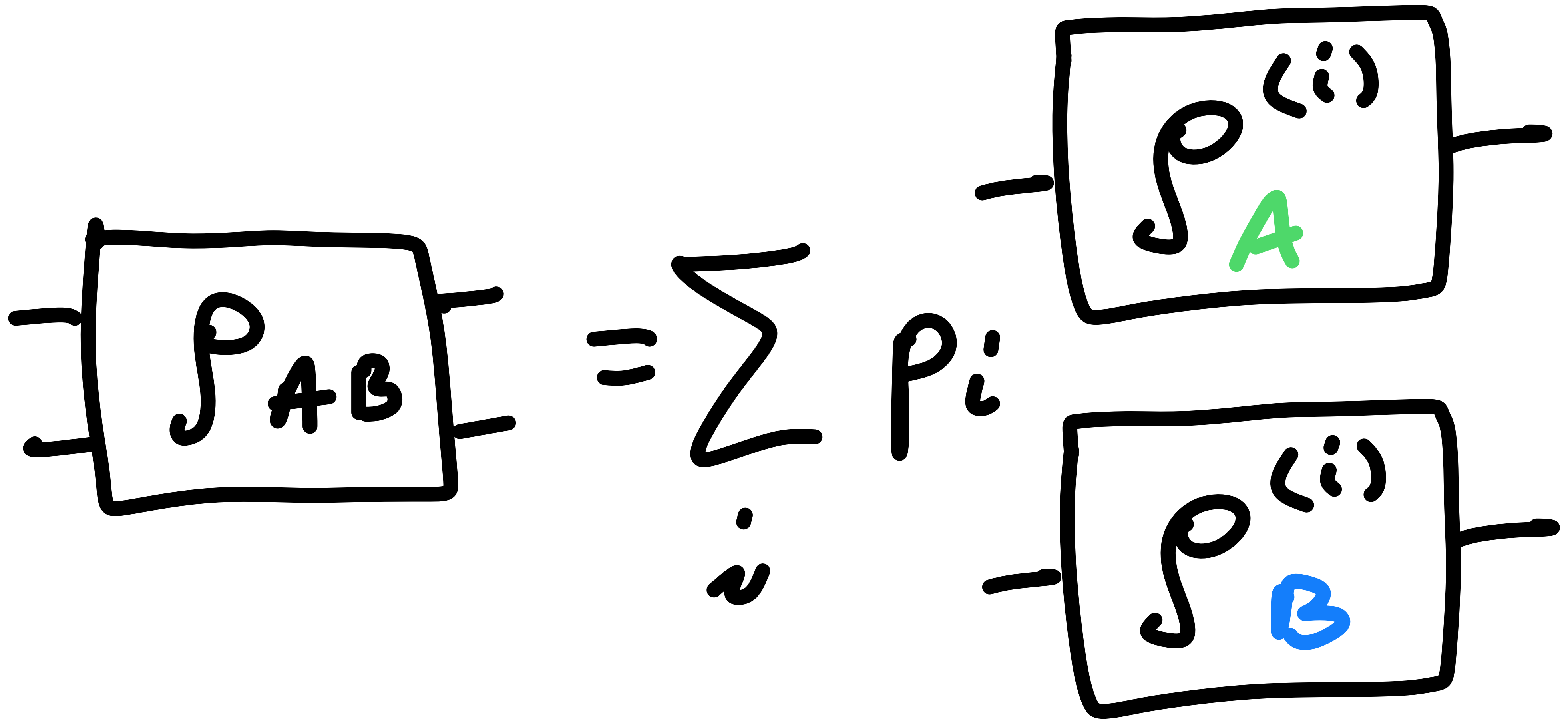
- In this case, the set of quantum states is given by bipartite matrices:

$$\{\rho_{AB} \in \mathcal{M}_d \otimes \mathcal{M}_d \cong \mathcal{M}_{d^2} \; : \; \rho \geq 0 \;\text{ and }\; \mathrm{Tr}\,\rho = 1\}$$



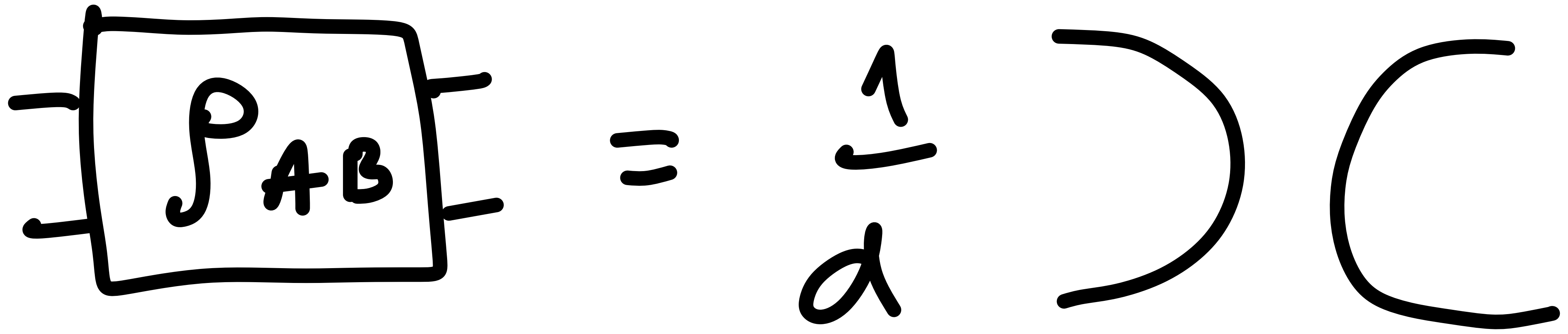
Two examples

- Separable states are bipartite quantum states which can be written as convex combinations of product states $\rho_A \otimes \rho_B$



- Entangled states are the non-separable states. The most important example is the maximally entangled state

$$\omega = \frac{1}{d} \sum_{i,j=1}^{d} |ii\rangle\langle jj|$$

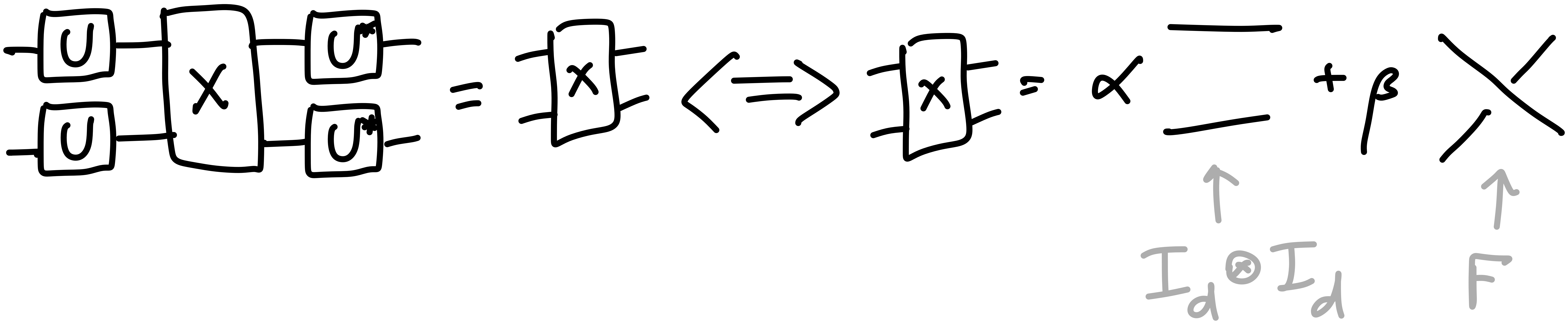


(Full) Unitary symmetry in the bipartite case

Theorem. Let $X \in \mathcal{M}_d \otimes \mathcal{M}_d$ be a bipartite operator. Then

$$\forall U \in \mathcal{U}_d, (U \otimes U)X(U \otimes U)^* = X \iff X = \alpha I_{d^2} + \beta F \text{ for } \alpha, \beta \in \mathbb{C}$$

where the (unitary) flip operator is defined by $F\, x \otimes y = y \otimes x$.

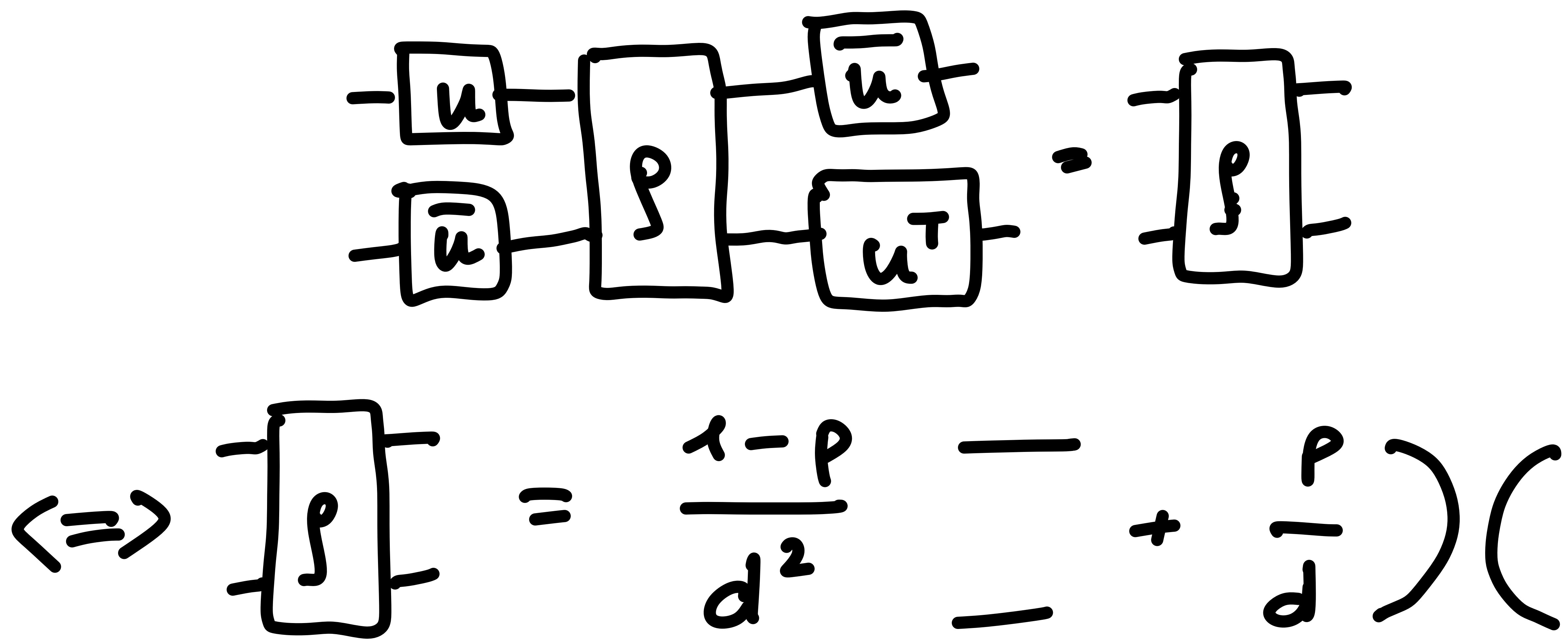


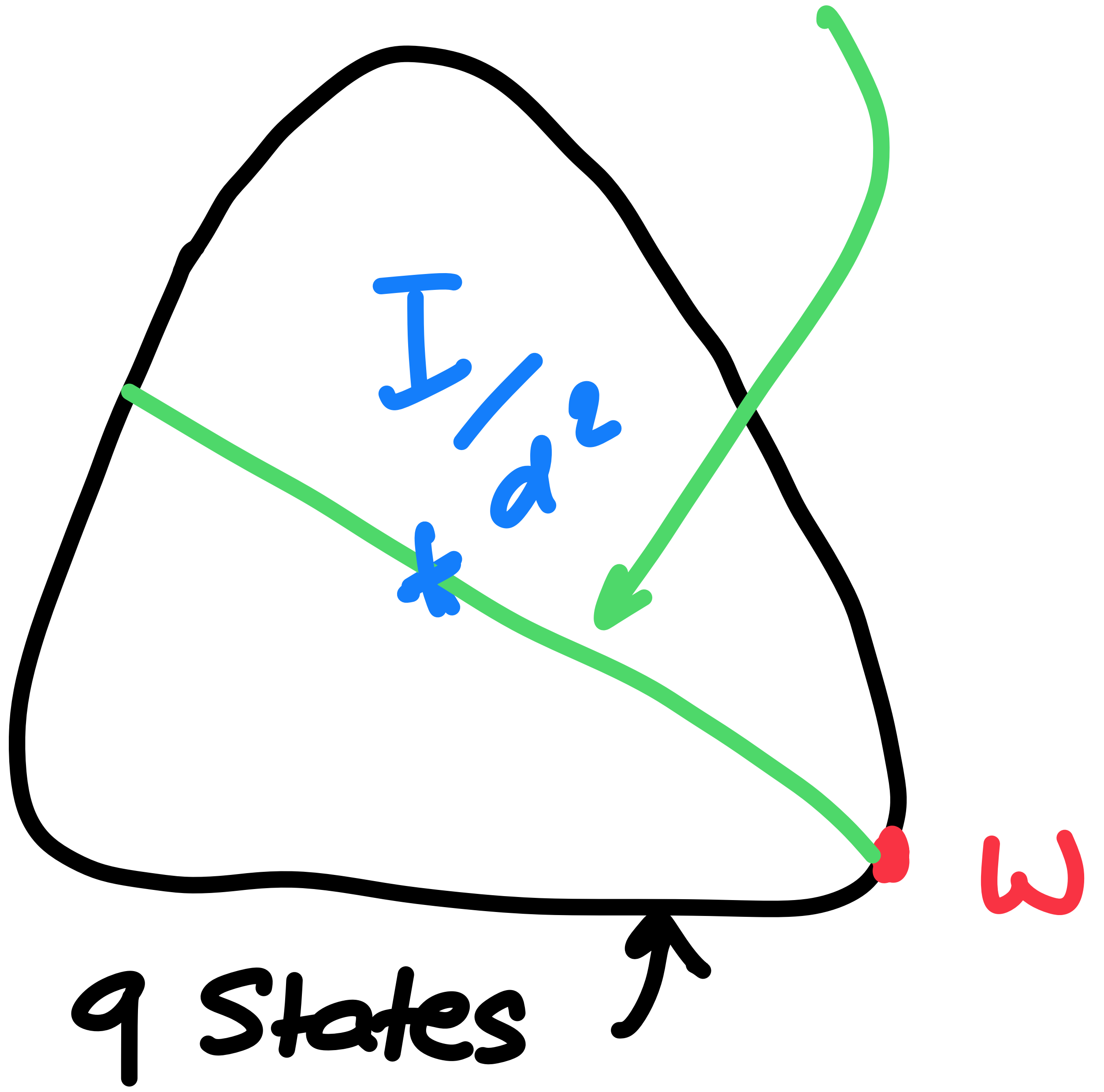
Isotropic quantum states

Theorem. Let $\rho \in \mathcal{M}_d \otimes \mathcal{M}_d$ be a bipartite density matrix. Then

$$\forall U \in \mathcal{U}_d, (U \otimes \bar{U})\rho U^* \otimes U^\top = \rho \iff \rho = (1-p)\frac{I}{d^2} + p\omega \text{ for } p \in [-1/(d^2-1), 1]$$

i.e. ρ must be a convex combination of the maximally mixed state and the maximally entangled state. Such quantum states are called isotropic.



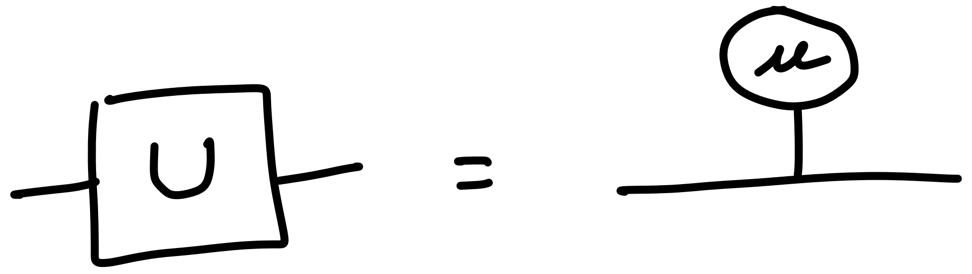


Diagonal unitary / orthogonal symmetry

The diagonal subgroup

- Since requiring the full unitary symmetry yields matrices (resp. quantum states) with only 2 parameters, we shall consider the much smaller subgroups

$$\mathscr{DU}_d := \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_d}) \ : \ \theta \in \mathbb{R}^d\}$$

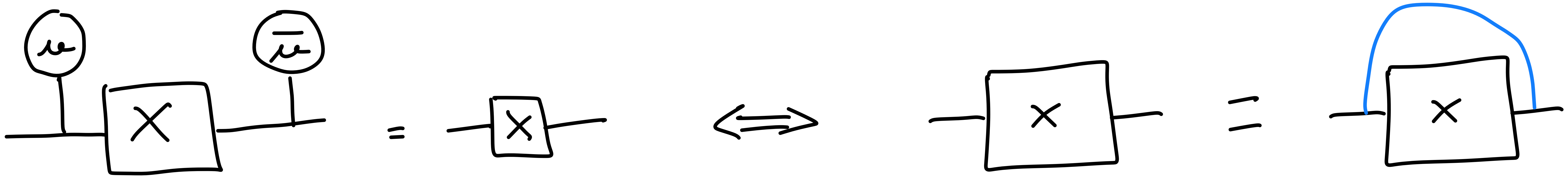


$$\mathscr{DO}_d := \{\mathrm{diag}(\epsilon_1, \ldots, \epsilon_d) \ : \ \epsilon \in \{\pm 1\}^d\}$$

$$u_{ij} = \delta_{ij} \cdot \mu_i$$

- In the case of a single tensor factor, we have

$$\forall U \in \mathscr{DU}_d, \quad UXU^* = X \iff X = \mathrm{diag}(X)$$

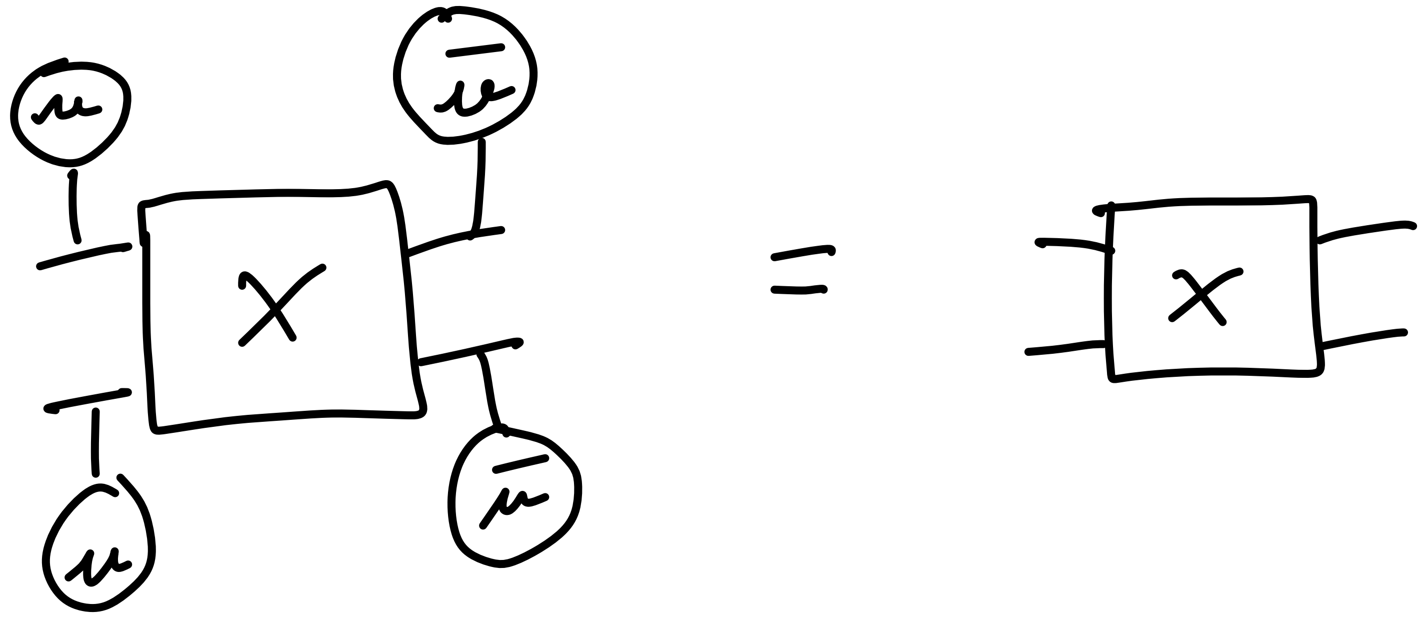


Diagonally symmetric bipartite matrices

Definition. A bipartite matrix $X \in \mathscr{M}_d \otimes \mathscr{M}_d$ is called:

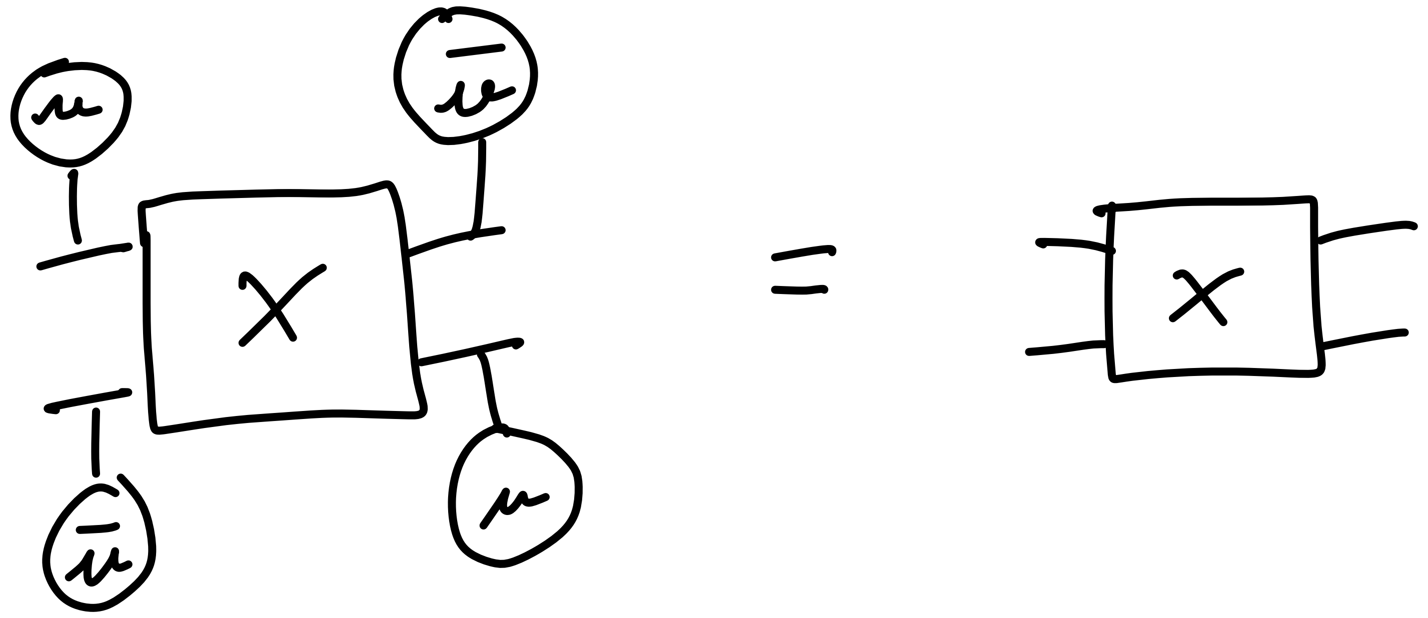
- LDUI (local diagonal unitary invariant) if

$$\forall U \in \mathscr{DU}_d, \quad (U \otimes U)X(U^* \otimes U^*) = X$$



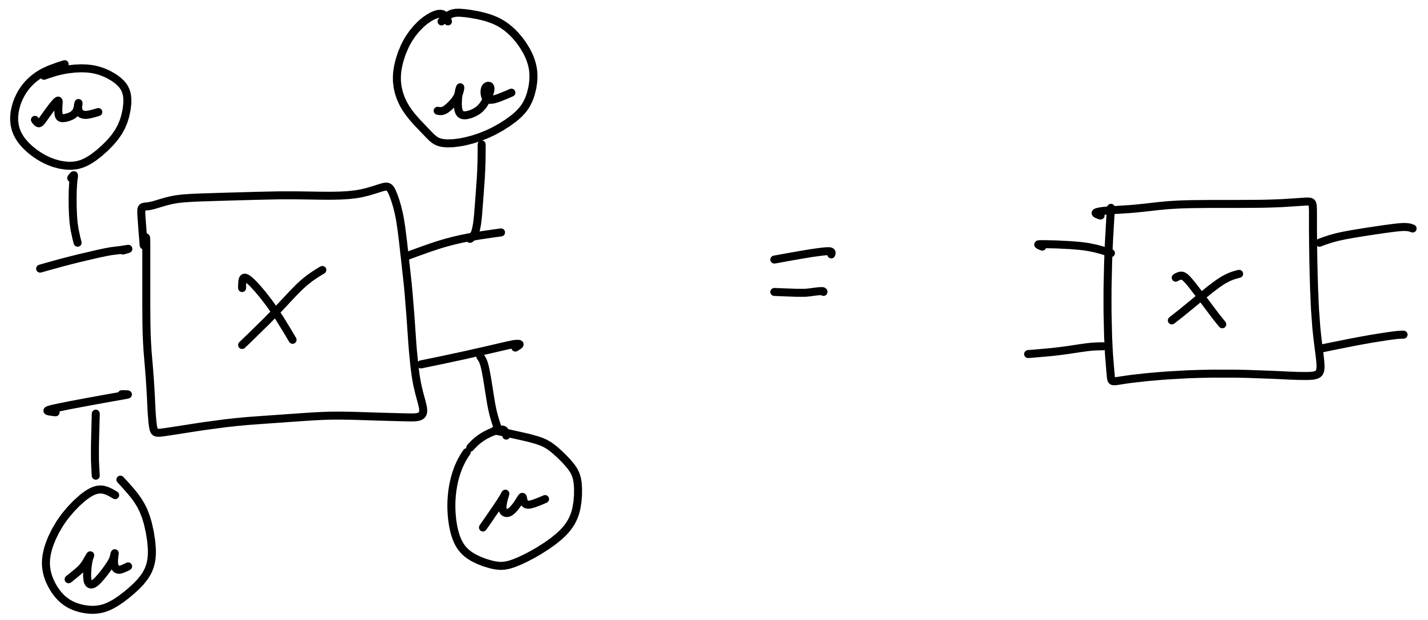
- CLDUI (conjugate LDUI) if

$$\forall U \in \mathscr{DU}_d, \quad (U \otimes \bar{U})X(U^* \otimes U^\top) = X$$



- LDOI (local diagonal orthogonal invariant) if

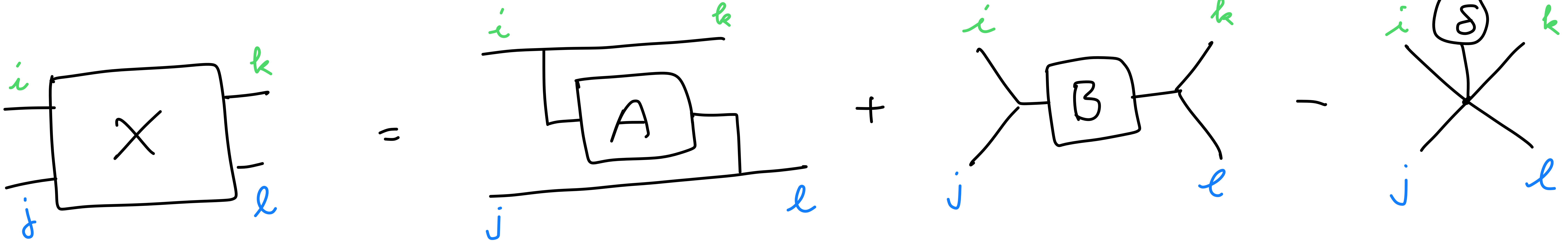
$$\forall U \in \mathscr{DO}_d, \quad (U \otimes U)X(U^\top \otimes U^\top) = X$$

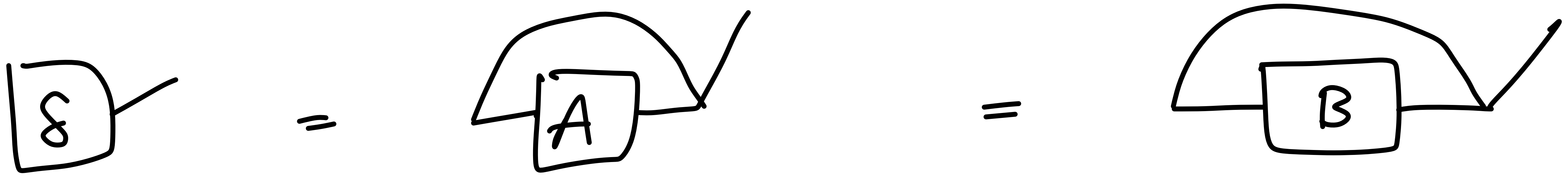


Characterisation theorem — CDLUI case

Theorem. A matrix $X \in \mathscr{M}_d \otimes \mathscr{M}_d$ is CLDUI iff there exist matrices $A, B \in \mathscr{M}_d$ having the same diagonal $\operatorname{diag} A = \operatorname{diag} B =: \delta \in \mathbb{C}^d$ such that

$$X_{ij,kl} = \mathbf{1}_{i=k,j=l} A_{ij} + \mathbf{1}_{i=j,k=l} B_{ik} - \mathbf{1}_{i=j=k=l} \delta_i$$





Characterization theorem - LDUI and LDOI

Theorem. A matrix $X \in \mathscr{M}_d \otimes \mathscr{M}_d$ is LDUI iff there exist matrices $A, C \in \mathscr{M}_d$ having the same diagonal $\operatorname{diag} A = \operatorname{diag} C =: \delta \in \mathbb{C}^d$ such that

$$X_{ij,kl} = \mathbf{1}_{i=k,j=l} A_{ij} + \mathbf{1}_{i=l,j=k} C_{ij} - \mathbf{1}_{i=j=k=l} \delta_i$$

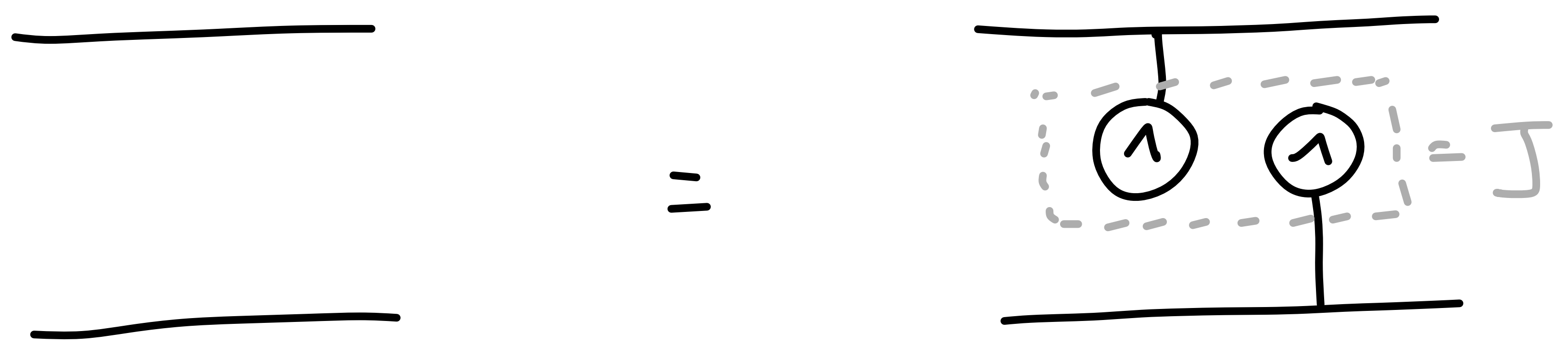
Theorem. A matrix $X \in \mathscr{M}_d \otimes \mathscr{M}_d$ is LDOI iff there exist matrices $A, B, C \in \mathscr{M}_d$ having the same diagonal $\operatorname{diag} A = \operatorname{diag} B = \operatorname{diag} C =: \delta \in \mathbb{C}^d$ such that

$$X_{ij,kl} = \mathbf{1}_{i=k,j=l} A_{ij} + \mathbf{1}_{i=j,k=l} B_{ik} + \mathbf{1}_{i=l,j=k} C_{ij} - 2\mathbf{1}_{i=j=k=l} \delta_i$$

Three examples

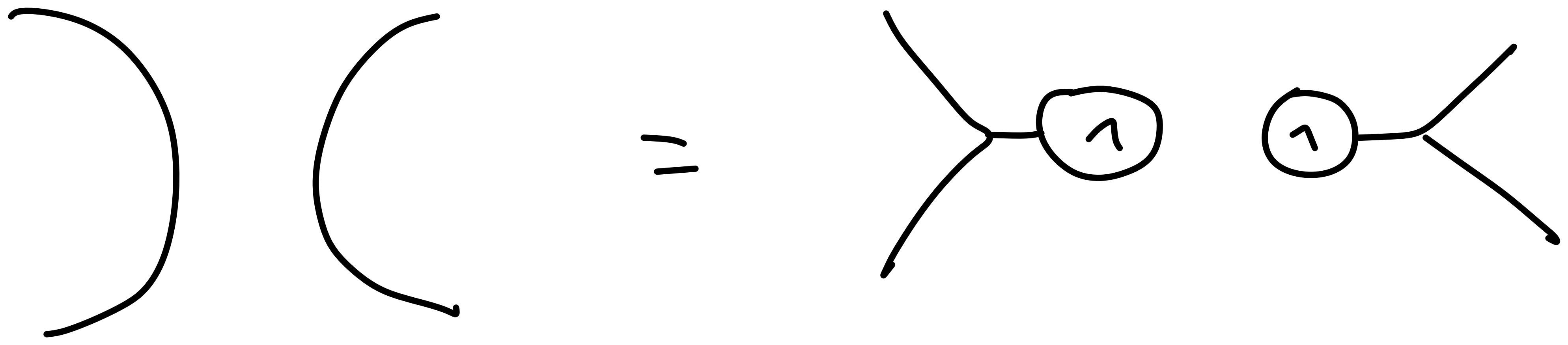
- The identity matrix is CLDUI with

  $A = J_d, B = I_d$



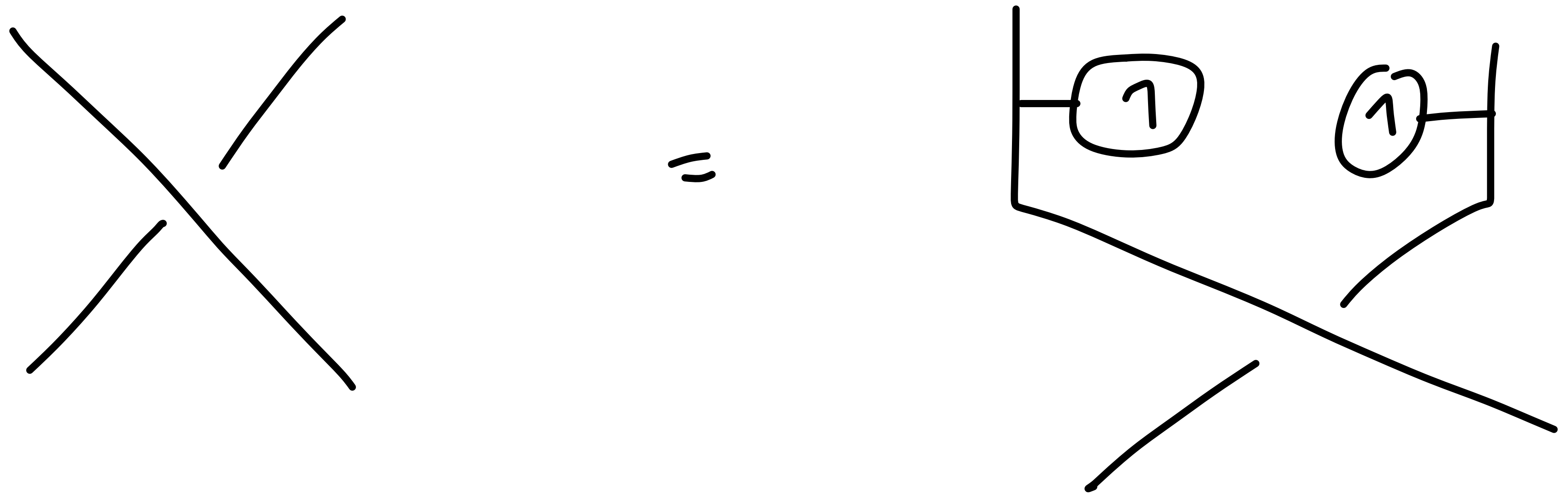
- The maximally entangled state is

  CLDUI with $A = I_d, B = J_d$



- The flip operator is LDUI with

  $A = I_d, B = J_d$



Symmetric bipartite PSD operators

Properties of symmetric operators

Theorem. A bipartite LDOI operator $X = X_{A,B,C}$ is

- self-adjoint iff A is real and B, C are self-adjoint
- positive semidefinite iff the following three conditions hold:
  1. A is entry-wise non-negative ($A_{ij} \geq 0$ for all i, j)
  2. B is positive semidefinite
  3. $A_{ij}A_{ji} \geq |C_{ij}|^2$ for all i, j

Note that LDUI operators correspond to B diagonal, and CLDUI operators correspond to C diagonal.

Separability for diagonal symmetric matrices

Theorem. A bipartite CLDUI operator of the form $X = X_{A,A} \in \mathscr{M}_d(\mathbb{C}) \otimes \mathscr{M}_d(\mathbb{C})$ is separable iff the matrix A is completely positive, i.e.

$$\exists R \in \mathscr{M}_{d\times k}(\mathbb{R}_+) \quad \text{s.t.} \quad A = RR^\top .$$

The columns of the non-negative square root R give the separable decomposition

$$X = \sum_{i=1}^{k} |r_i\rangle\langle r_i| \otimes |\bar{r}_i\rangle\langle \bar{r}_i| .$$

In this case, $A = \sum_{i=1}^{k} \Big| |r_i|^2 \Big\rangle \Big\langle |r_i|^2 \Big|$. In dimensions $d \leq 4$ completely positive matrices are precisely positive semidefinite matrices with non-negative entries.

PCP and TCP matrices

Definition. A pair of matrices (A, B) is called pairwise completely positive (PCP) if there exist matrices $V, W \in \mathcal{M}_{d\times k}(\mathbb{C})$ such that

$$A = (V \odot \bar{V})(W \odot \bar{W})^* \quad \text{and} \quad B = (V \odot W)(V \odot W)^* .$$

A triple (A, B, C) is called triplewise completely positive (TCP) if, additionally, $C = (V \odot \bar{W})(V \odot \bar{W})^*$.

Theorem. A (C)LDUI matrix $X_{A,B}$ is separable iff the pair (A, B) is PCP. An LDOI matrix $X_{A,B,C}$ is separable iff the triple (A, B, C) is TCP.

Application: the PPT² conjecture

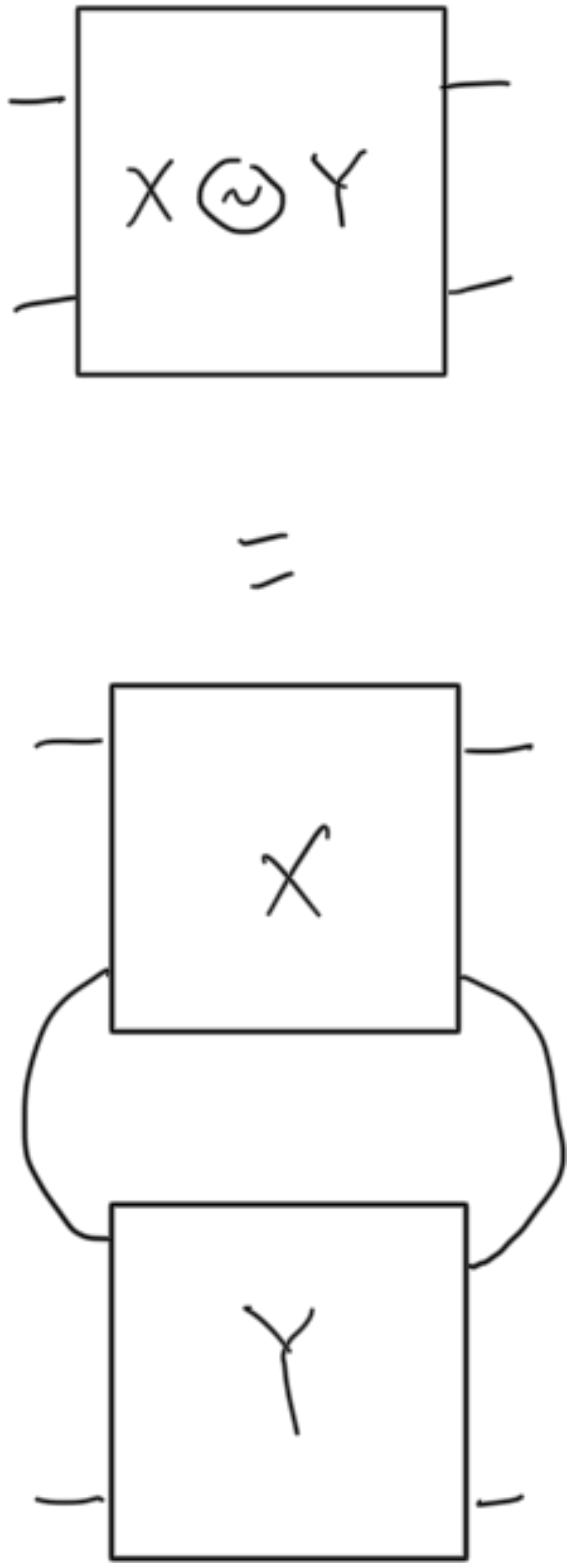
Conjecture. The link product of two PPT matrices is separable.

Recall that a (C)LDUI matrix $X_{A,B}$ is PPT (positive partial transpose) iff A is entrywise non-negative, B is PSD, and $\forall i, j, A_{ij}A_{j,i} \geq |B_{ij}|^2$.

Theorem. The PPT² conjecture holds for (C)LDUI matrices.

Proposition. Let $X_{A,B}$ be a PPT (C)LDUI matrix. If B is diagonally dominant (i.e. $\forall i, \quad A_{ii} = B_{ii} \geq \sum_{j \neq i} |B_{ij}|$), then (A, B) is PCP ($\iff \quad X_{A,B}$ is separable).

The proof of the proposition relies on the notion of factor width.



Take home slide

- Multipartite quantum states that are symmetric with by conjugation with diagonal unitary (resp. orthogonal) matrices form a rich, interesting class.
- The CLDUI class is parametrised by two matrices A, B having a common diagonal. This class contains all classical (diagonal) states and the maximally entangled state.
- A CLDUI matrix $X_{A,B}$ is PSD iff A is entry-wise non-negative and B is PSD.
- A CLDUI matrix $X_{A,B}$ is separable iff the pair (A, B) is pairwise completely positive (PCP). This is a generalisation of completely positive matrices.
- (C)LDUI matrices satisfy the PPT^2 conjecture; the proof uses the notion of factor width and its generalisation to the PCP setting.