Operator - valued free probability and tensor flattenings

a numerical experiment

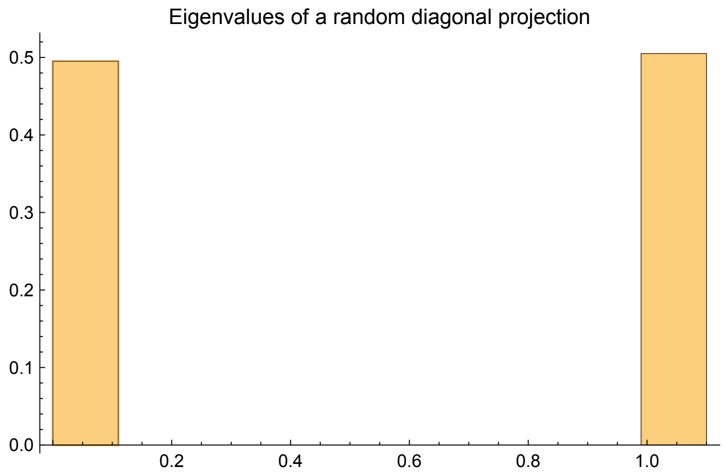
In[433]:=

```
ClearAll["Global`*"]
```

In[434]:=

```
d = 3000;
diagP = DiagonalMatrix[1.0 * Table[RandomInteger[{0, 1}], d]];
Histogram[Eigenvalues[diagP], {0.11}, "Probability",
 PlotLabel → "Eigenvalues of a random diagonal projection"]
```

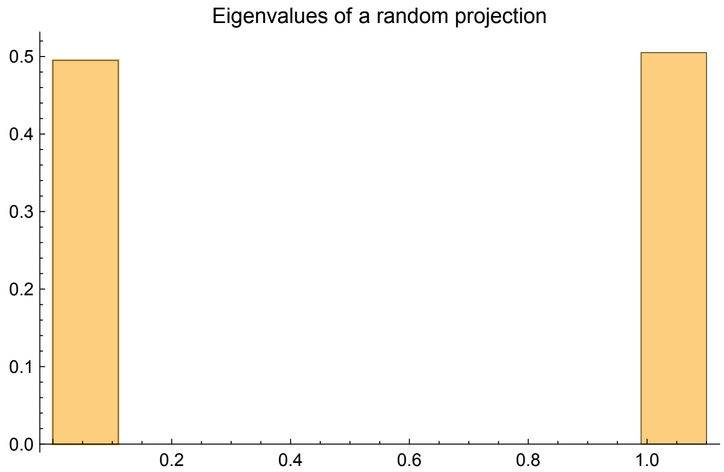
Out[436]=



In[437]:=

```
UP = RandomVariate[CircularUnitaryMatrixDistribution[d]];
matP = UP.diagP.ConjugateTranspose[UP];
Histogram[Eigenvalues[matP] // Chop, {0.11},
 "Probability", PlotLabel → "Eigenvalues of a random projection"]
```

Out[439]=

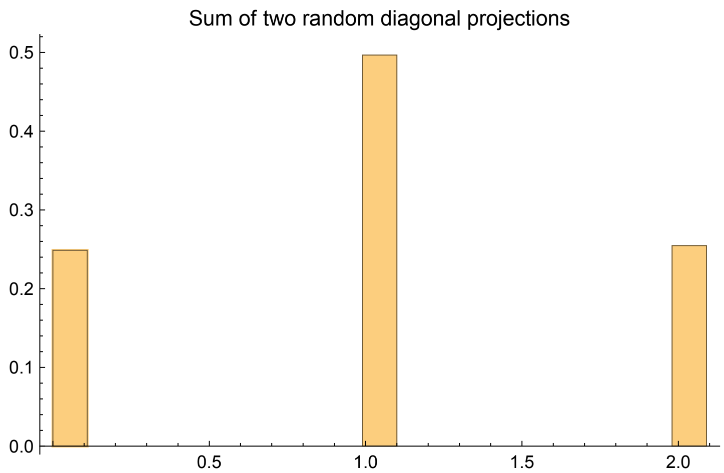


Sum of two random diagonal projectors

In[440]:=

```
diagQ = DiagonalMatrix[1.0 * Table[RandomInteger[{0, 1}], d]];
sumClassical = diagP + diagQ;
Histogram[Eigenvalues[sumClassical] // Chop, {0.11},
 "Probability", PlotLabel → "Sum of two random diagonal projections"]
```

Out[442]=

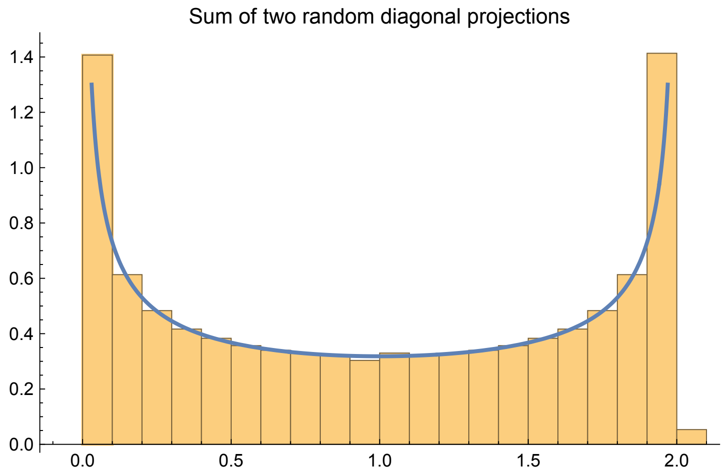


Sum of two random projectors

In[443]:=

```
UQ = RandomVariate[CircularUnitaryMatrixDistribution[d]];
matQ = UQ.diagQ.ConjugateTranspose[UQ];
sumFree = matP + matQ;
Show[Histogram[Eigenvalues[sumFree] // Chop, 20,
  "PDF", PlotLabel → "Sum of two random diagonal projections",
  PlotRange → {{-0.1, 2.1}, Automatic}], Plot[1 / (Pi Sqrt[x (2 - x)]),
  {x, 0, 2}, PlotStyle → {Thick}, PlotRange → {Automatic, {-0.1, 1.3}}]]
```

Out[446]=



Above, the curve in blue is the probability density function of the **arcsine distribution** given by

$$\frac{1}{\pi\sqrt{x(2-x)}}\, 1_{(0,2)}(x)\, d\,x$$

Voiculescu’s free probability theory

Theorem. Independent, unitarily invariant random matrices are asymptotically free.

In the examples above, the (diagonal or not) random projections have limiting distribution

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$$

When summing the diagonal projections, we obtain the **classical additive convolution**

$$\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) * \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) = \frac{1}{4}\delta_0 + \frac{1}{2}\delta_1 + \frac{1}{4}\delta_2$$

When summing the randomly rotated projections, we obtain the **free additive convolution**

$$\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) \boxplus \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) = \frac{1}{\pi\sqrt{x\,(2-x)}}\, \mathbf{1}_{(0,2)}(x)\, dx$$

Unitary invariance hypothesis

We shall now consider the following two **block matrices**

$$\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \text{ and } \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}$$

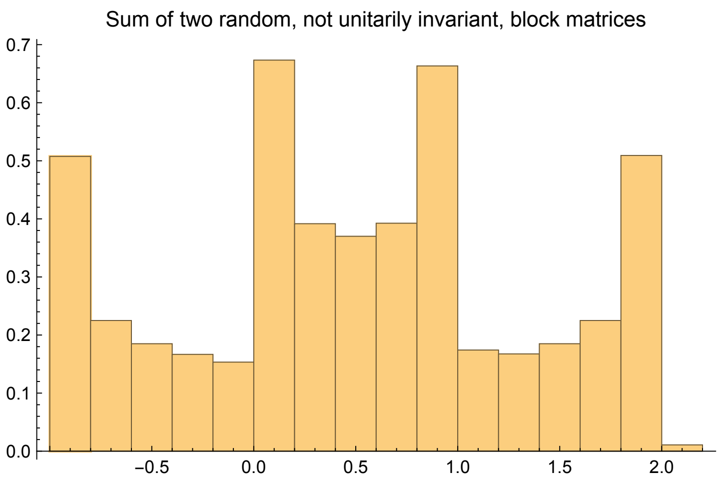
Clearly, they are independent, but **not unitarily invariant** (although P and Q are). The first one converges, as $d \to \infty$, to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ while the second one converges to $\frac{1}{4}\delta_{-1} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_1$. We consider their their sum

$$X = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix}$$

In[447]:=

```
blockP = ConstantArray[0, {2 d, 2 d}];
blockP[[1 ;; d, 1 ;; d]] = matP;
blockP[[d + 1 ;; 2 d, d + 1 ;; 2 d]] = matP;
blockQ = ConstantArray[0, {2 d, 2 d}];
blockQ[[d + 1 ;; 2 d, 1 ;; d]] = matQ;
blockQ[[1 ;; d, d + 1 ;; 2 d]] = matQ;
X = blockP + blockQ;
Histogram[Eigenvalues[X] // Chop, 20, "PDF",
 PlotLabel → "Sum of two random, not unitarily invariant, block matrices"]
```

Out[454]=



Note, however, that if we rotate the second matrix by a random unitary matrix of full size ($2d$)

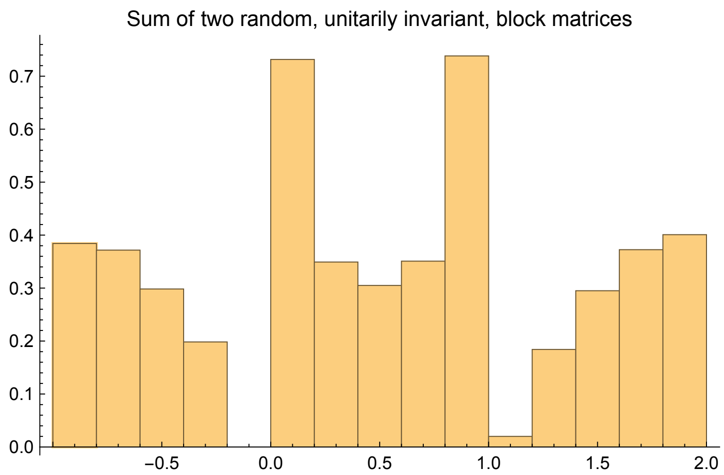
$$Y = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} + U \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} U^*$$

we obtain a different eigenvalue density

In[455]:=

```
largeU = RandomVariate[CircularUnitaryMatrixDistribution[2 d]];
blockQUI = largeU.blockQ.ConjugateTranspose[largeU];
Y = blockP + blockQUI;
Histogram[Eigenvalues[Y] // Chop, 20, "PDF",
 PlotLabel → "Sum of two random, unitarily invariant, block matrices"]
```

Out[458]=



Above, the summands satisfy the assumptions of Voiculescu's theorem, to the limiting probability distribution is a free additive convolution:

$$\mu = \left(\frac{1}{2}\,\delta_0 + \frac{1}{2}\,\delta_1\right) \boxplus \left(\frac{1}{4}\,\delta_{-1} + \frac{1}{2}\,\delta_0 + \frac{1}{4}\,\delta_1\right)$$

The last two plots are visually different, the limiting distributions are not identical. This can also be seen by looking at the following mixed moments which differ in the two situations

In[459]:=

```
1 / (2 d) Tr[blockP.blockQ.blockP.blockQ] // N // Chop
1 / (2 d) Tr[blockP.blockQUI.blockP.blockQUI] // N // Chop
```

Out[459]=

0.190509 → Later we shall see that this $\xrightarrow{d \to \infty} \frac{3}{16}$

Out[460]=

0.127802 → Later we shall see that this $\xrightarrow{d \to \infty} \frac{1}{8}$

In the first situation, the random matrices are not unitarily invariant, and one cannot apply Voiculescu's theorem. As it turns out, they are not asymptotically free, but only **operator-valued** asymptotically **free**. This is the topic of the **first lecture**, introducing the basics of operator-valued free probability and the applications to block random matrices.

Independence hypothesis

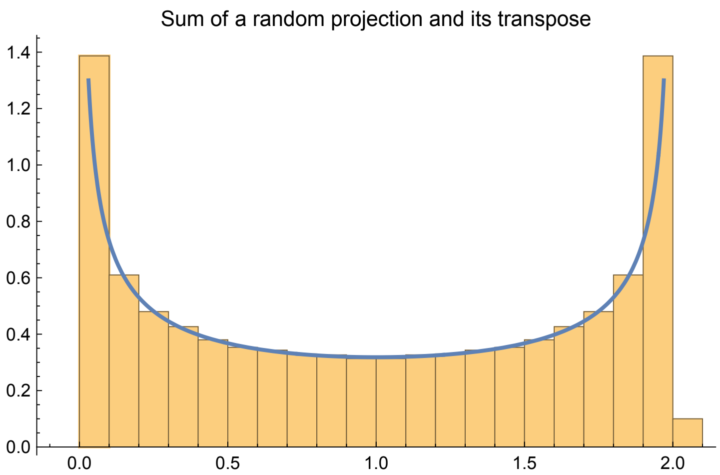
Recall that $P = U\,\mathrm{diag}\,(p_1, \ldots, p_d)\,U^*$ is a random, unitarily invariant, projection of rank $d/2$. The matrices P and P^T are clearly not independent, since they have the same entries, up to permutation. However, Mingo and Popa have shown the following result.

Theorem. A unitarily invariant random matrix is asymptotically free from its transpose.

In[461]:=

```
sumTranspose = matP + Transpose[matP];
Show[Histogram[Eigenvalues[sumTranspose] // Chop, 20, "PDF",
  PlotLabel → "Sum of a random projection and its transpose",
  PlotRange → {{-0.1, 2.1}, Automatic}], Plot[1 / (Pi Sqrt[x (2 - x)]),
  {x, 0, 2}, PlotStyle → {Thick}, PlotRange → {Automatic, {-0.1, 1.3}}]]
```

Out[462]=



So asymptotic freeness can hold even in the absence of the independence condition. We shall explore this phenomenon during the **second lecture**, when we shall generalize Mingo and Popa's result to **flattenings of random tensors**.

LECTURE I - Introduction to operator-valued free probability

Definition A (scalar) non-commutative probability space is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a $*$-algebra and $\varphi: \mathcal{A} \to \mathbb{C}$ is a unital linear form.

An operator-valued nc. prob. space is a triple $(\mathcal{A}, \mathcal{B}, \mathbb{E})$, where $1 \in \mathcal{B} \subseteq \mathcal{A}$ are unital $*$-algebras and $\mathbb{E}: \mathcal{A} \to \mathcal{B}$ is a linear map, called conditional expectation:

- $\mathbb{E}(1) = 1$
- $\mathbb{E}(bab') = b\,\mathbb{E}(a)\,b' \quad \forall a \in \mathcal{A},\ b, b' \in \mathcal{B}$

Definition Subalgebras $\mathcal{B} \subseteq A_1, \ldots, A_k \subseteq \mathcal{A}$ are called $\mathcal{B}$-free if

$$\mathbb{E}(\bar{a}_1 \bar{a}_2 \cdots \bar{a}_r) = 0$$

whenever $a_i \in A_{j(i)}$ with $j(1) \neq j(2) \neq \cdots \neq j(r)$

(for centered $\bar{a} := a - \mathbb{E}(a) \cdot 1$)

Example Let $(\mathcal{A}, \varphi)$, $\varphi: \mathcal{A} \to \mathbb{C}$ be a usual $\mathbb{C}$-nc probability space. Consider 3 different operator-valued nc prob spaces for $M_n(\mathcal{A}) \simeq M_n(\mathbb{C}) \otimes \mathcal{A}$:

- $\mathbb{E}_1\left((a_{ij})_{i,j\in[n]} \right) = \frac{1}{n}\sum_i \varphi(a_{ii}) \cdot I_n \in \mathbb{C} I_n \simeq \mathbb{C}$

  $\mathbb{E}_1 : M_n(\mathcal{A}) \longrightarrow \mathbb{C} I_n$

- $\mathbb{E}_2\left((a_{ij}) \right) = \left(\delta_{ij}\, \varphi(a_{ij}) \right)_{i,j\in[n]} \in D_n$

  diagonal $n \times n$ matrices

  $\mathbb{E}_2 : M_n(\mathcal{A}) \longrightarrow D_n$

- $\mathbb{E}_3\left((a_{ij}) \right) = \left(\varphi(a_{ij}) \right)_{ij\in[n]}$

  $\mathbb{E}_3 = \mathrm{id}_n \otimes \varphi : M_n(\mathcal{A}) \longrightarrow M_n(\mathbb{C})$

Theorem If $A_1, \dots, A_k \subseteq \mathcal{A}$ are free, then $M_n(\mathbb{C}) \otimes A_1, \dots, M_n(\mathbb{C}) \otimes A_k$ are $M_n(\mathbb{C})$-free in $(M_n(\mathcal{A}), M_n(\mathbb{C}), \mathbb{E}_3)$

(but, in general, not D_n-free nor $\mathbb{C}$-free)

How to use (op-val) freeness:

- x, y free, then

$$\varphi(xy) = \varphi\left((\bar{x} + \varphi(x)1)\cdot(\bar{y} + \varphi(y)\cdot 1) \right)$$

$$= \underbrace{\varphi(\bar{x}\bar{y})}_{\substack{\| \\ 0 \\ \text{(freeness)}}} + \underbrace{\varphi(\bar{x})}_{\substack{\| \\ 0}} \varphi(y) + \varphi(x) \underbrace{\varphi(\bar{y})}_{\substack{\| \\ 0}} + \varphi(x)\varphi(y)$$

- If $x, y \in \mathcal{A}$ are $\mathbb{C}$-free, then

$$\varphi(xyxy) = \varphi(x^2)\varphi(y)^2 + \varphi(x)^2\varphi(y^2) - \varphi(x)^2\varphi(y)^2$$

↳ different than classical probability

- if $x, y \in \mathcal{A}$ are $\mathcal{B}$-free, then

$$\begin{aligned} \mathbb{E}(xyxy) = {} & \mathbb{E}(x\,\mathbb{E}(y)\,x)\,\mathbb{E}(y) + \\ & + \mathbb{E}(x)\,\mathbb{E}(y\,\mathbb{E}(x)\,y) \\ & - \mathbb{E}(x)\,\mathbb{E}(y)\,\mathbb{E}(x)\,\mathbb{E}(y) \end{aligned}$$

↳ remember that $\mathbb{E}$ is $\mathcal{B}$-valued so $\mathbb{E}(x), \mathbb{E}(y)$ do not commute in general!

Back to the example in the numerical experiment

Recall that P_d, Q_d are indep., unitarily invariant projections of rank $d/2$. So

$$(P_d, Q_d) \xrightarrow[d \to \infty]{} (p, q) \qquad \begin{array}{l} p, q \text{ free proj} \\ \varphi(p) = \varphi(q) = \frac{1}{2} \end{array}$$

Consider now

$$\tilde{P}_d := \begin{pmatrix} P_d & 0 \\ 0 & P_d \end{pmatrix}, \quad \tilde{Q}_d := \begin{pmatrix} 0 & Q_d \\ Q_d & 0 \end{pmatrix} \in M_{2d}(\mathbb{C})^{sa}$$

and $\tilde{R}_d := U \cdot \tilde{Q}_d U^*$ for $U \in \mathcal{U}(2d)$ Haar-distributed, independent of P_d, Q_d

If we denote by $\mu_X := \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(X)}$ the empirical eigenvalue distribution of a matrix X, then we have

$$\mu_{\tilde{P}_d} \xrightarrow[d\to\infty]{} \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \qquad \left.\begin{array}{l}\mu_{\tilde{Q}_d}\\ \mu_{\tilde{R}_d}\end{array}\right\} \xrightarrow[d\to\infty]{} \frac{1}{4}\delta_{-1} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_1$$

→ note that $\tilde{P}_d = P \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\tilde{Q}_d = Q \otimes \begin{pmatrix}0&1\\1&0\end{pmatrix}$

so $\operatorname{spec}(\tilde{P}_d) = \operatorname{spec}(P_d)^{\times 2}$ and $\operatorname{spec}(\tilde{Q}_d) = \pm \operatorname{spec}(Q_d)$

Fact 1

By Voiculescu's theorem, $\tilde{P}_d$ and $\tilde{R}_d$ are asympt. $\mathbb{C}$-free, so:

$$\frac{1}{2d}\mathbb{E}\operatorname{Tr}(\tilde{P}_d \tilde{R}_d \tilde{P}_d \tilde{R}_d) \xrightarrow[d\to\infty]{} \varphi(prpr)$$ where p, r are free elements with resp distributions

$$p \sim \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \quad \text{and} \quad r \sim \frac{1}{4}\delta_{-1} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_1$$

By the previous result:

$$\varphi(prpr) = \underbrace{\varphi(\overset{p}{\overbrace{p^2}})}_{=\frac{1}{2}}\underbrace{\varphi(r)^2}_{=0} + \underbrace{\varphi(p)^2}_{\left(\frac{1}{2}\right)^2}\underbrace{\varphi(r^2)}_{\frac{1}{2}} - \varphi(p)^2\underbrace{\varphi(r)^2}_{0}$$

$$= \frac{1}{8}$$

Fact 2

By the operator-valued freeness result, $\tilde{P}_d$ and $\tilde{Q}_d$ are asymptotically $M_2(\mathbb{C})$-free.

$$\mathbb{E}_d : M_{2d}(L^{\infty-}(\mathbb{P})) \to M_2(\mathbb{C})$$

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix} \mapsto \begin{pmatrix}\frac{1}{d}\mathbb{E}\operatorname{Tr}A & \frac{1}{d}\mathbb{E}\operatorname{Tr}B\\ \frac{1}{d}\mathbb{E}\operatorname{Tr}C & \frac{1}{d}\mathbb{E}\operatorname{Tr}D\end{pmatrix}$$

$$\mathbb{E}_d(\tilde{P}_d) \to \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbb{E}_d(\tilde{Q}_d) \to \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

By the computation for op-valued freeness:

$$\mathbb{E}_d(\tilde{P}_d \tilde{Q}_d \tilde{P}_d \tilde{Q}_d) \to \mathbb{E}(pqpq) \quad \text{for}$$

$M_2(\mathbb{C})$-free elements p, q

$$\text{so} \quad \mathbb{E}[pqpq] = \mathbb{E}[p \underbrace{\mathbb{E}[q]}_{\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} p] \underbrace{\mathbb{E}[q]}_{\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$+ \underbrace{\mathbb{E}[p]}_{\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathbb{E}[q \underbrace{\mathbb{E}[p]}_{\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} q]$$

$$- \underbrace{\mathbb{E}[p]\,\mathbb{E}[q]\,\mathbb{E}[p]\,\mathbb{E}[q]}_{\frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Compute $\mathbb{E}[p \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot p] = \lim_{d \to \infty} \mathbb{E}_d \begin{pmatrix} P_d & 0 \\ 0 & P_d \end{pmatrix}\begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}\begin{pmatrix} P_d & 0 \\ 0 & P_d \end{pmatrix}$

$$= \lim_{d \to \infty} \mathbb{E}_d \begin{pmatrix} 0 & P_d \\ P_d & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbb{E}[pqpq] = \frac{1}{8}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \varphi(pqpq) = \frac{1}{2} \operatorname{Tr} \mathbb{E}(pqpq) = \frac{3}{16}$$

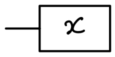
This matches the numerial experiment.

LECTURE II - Flattenings of random tensors

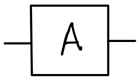
joint work with Camille Male and Stéphane Dartois, arXiv:2307.11439

1 Tensors and flattenings

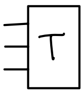
- scalars $\mathbb{C}$ N^0 param



- vectors $x \in \mathbb{C}^N$ N param. $(x_i)_{i=1}^N$



- matrices $A \in \mathcal{M}_N(\mathbb{C})$ N^2 param $(a_{ij})_{i,j\in[N]}$

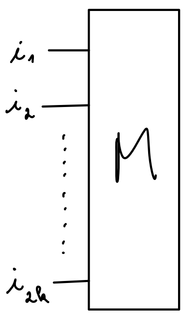


- 3-tensor $T \in \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ N^3 param

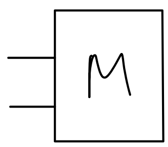
$$(T_{ijk})_{i,j,k\in[N]}$$

- 2k-tensor $M \in (\mathbb{C}^N)^{\otimes 2k}$

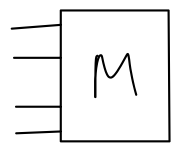
$$(M_{i_1,\dots,i_{2k}})_{i_1,\dots,i_{2k}\in[N]}$$



$k=1$



$k=2$

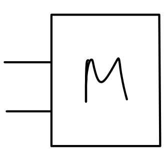


Flattening transforming a tensor into a matrix

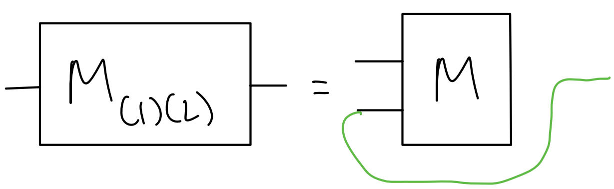
hard to study
no eigenvalues
multi-linear algebra

easy
canonical form
linear algebra

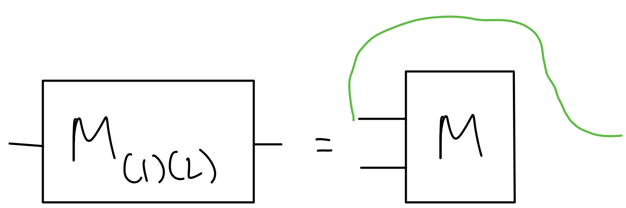
$k=1$



has two flattenings: $M_{(1)(2)}, M_{(12)} \in \mathcal{M}_N(\mathbb{C})$



and



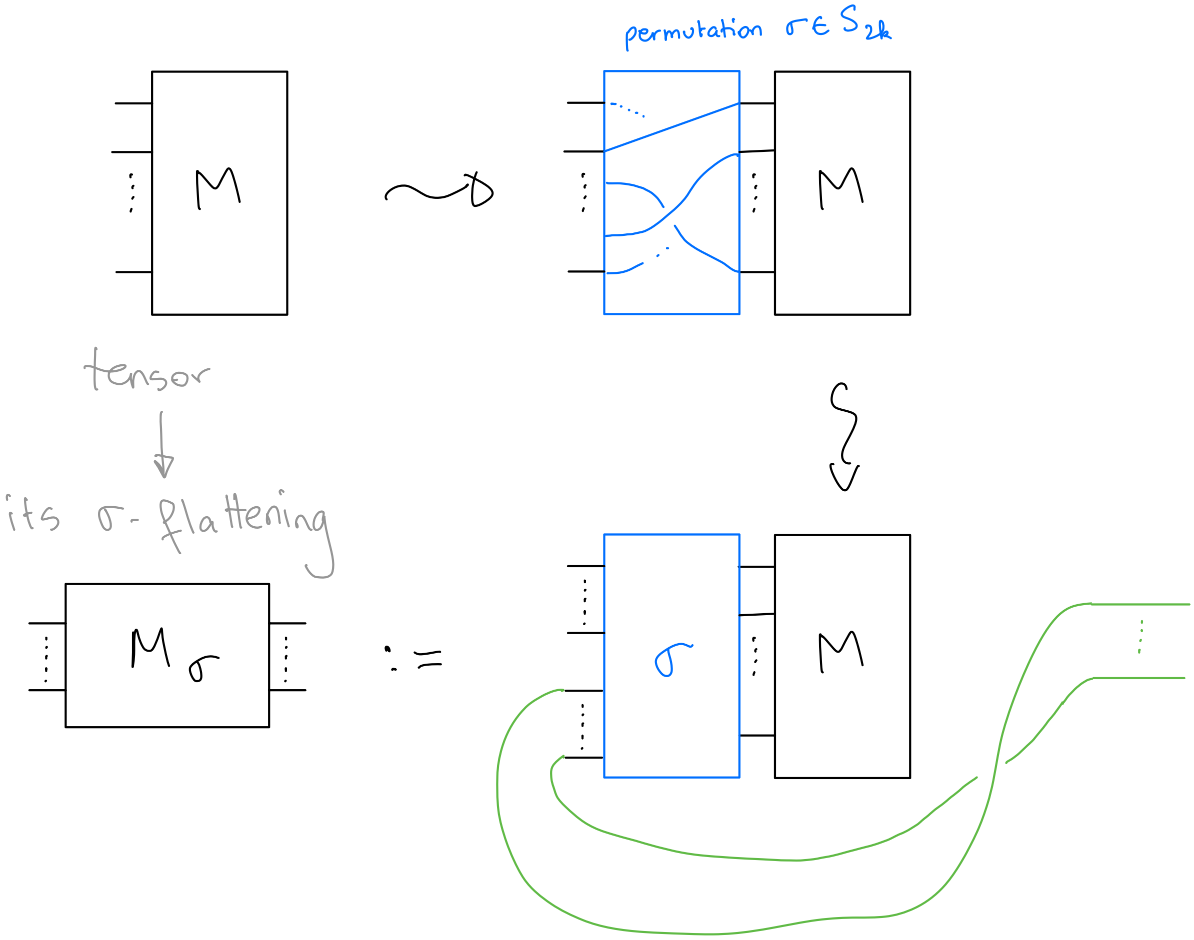
Using coordinates:

$$M_{(1)(2)}(i,j) = M_{i,j}$$
$$M_{(12)}(i,j) = M_{j,i}$$

$$\Rightarrow \quad M_{(12)} = M^{T}_{(1)(2)}$$

General case

$$M \in (\mathbb{C}^N)^{\otimes 2k}$$



$$(\mathbb{C}^N)^{\otimes 2k} \ni M \rightsquigarrow \{M_\sigma\}_{\sigma \in S_{2k}} \subseteq \mathcal{M}^{sa}_{N^k \times N^k}(\mathbb{C})$$

tensor $\to$ its flattenings

random M $\to$ dependent random mat.

2. Model of random tensors and main result

Hypothesis 1.2. *The tensor $M_N \in (\mathbb{C}^N)^{\otimes 2k}$ has i.i.d. entries distributed as a centered complex random variable m_N having finite moments of all orders (i.e. $\mathbb{E}\big[|m_N|^\ell\big] < \infty$ for all ℓ). Moreover, the following limits exist*

$$N^k \times \mathbb{E}\big[|m_N|^2\big] \underset{N\to\infty}{\longrightarrow} c > 0, \quad N^k \times \mathbb{E}\big[m_N^2\big] \underset{N\to\infty}{\longrightarrow} c' \in \mathbb{C}, \tag{1.2}$$

and for all non-negative integers ℓ_1, ℓ_2 such that $\ell_1 + \ell_2 > 2$

$$N^k \times \mathbb{E}\big[m_N^{\ell_1}\overline{m_N}^{\ell_2}\big] \underset{N\to\infty}{\longrightarrow} 0. \tag{1.3}$$

We call (c, c') the parameter of M_N.

1. *Let M_N be sampled according to the complex Ginibre ensemble, i.e. the entries of $\sqrt{N}M_N$ are distributed according to the standard complex Gaussian distribution. Then M_N satisfies Hypothesis 1.2 and its parameter is $(1, 0)$. A real Ginibre ensemble also satisfies the hypothesis with parameter $(1, 1)$.*

Example in the tensor case $(c = 1, c' = 0)$:

$N^{k/2} \cdot M_N(i_1, \dots, i_{2k})$ iid $\mathcal{N}_{\mathbb{C}}(0, 1)$

The main theorem

Theorem 2.15. *Let M_N be a random tensor satisfying Hypothesis 1.2 with parameter (c, c'). Then the collection of flattenings of M_N converges in $\mathfrak{S}_k^*$-distribution to a centered $\mathfrak{S}_k$-circular family $\mathbf{m} = (m_\sigma)_{\sigma \in \mathfrak{S}_{2k}}$, in some space $(\mathcal{A}, \mathbb{C}\mathfrak{S}_k, \mathcal{E})$. For all $\eta, \eta' \in \mathfrak{S}_k$, we denote $\eta \sqcup \eta' \in \mathfrak{S}_{2k}$ the permutation obtained by gluing the actions of η and η' on the first k and the last k elements of $[2k]$; it is formally defined by*

$$(\eta \sqcup \eta')(i) := \begin{cases} \eta(i) & \text{if } i \in [k] \\ \eta'(i-k)+k & \text{otherwise.} \end{cases} \tag{2.8}$$

We also denote by $\tau \in \mathfrak{S}_{2k}$ the permutation swapping the first and the last k elements of $[2k]$; it is given by

$$\tau(i) \quad : \quad \begin{cases} i \in [k] & \mapsto & i+k \in [2k] \setminus [k] \\ i \in [2k] \setminus [k] & \mapsto & i-k \in [k] \end{cases}$$

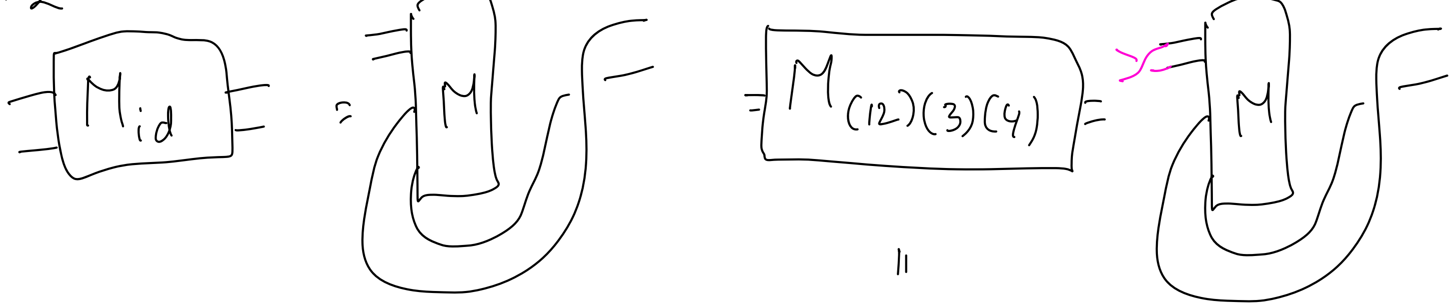
The $\mathfrak{S}_k$-covariance of $\mathbf{m}$ is given by the following equalities: $\forall \sigma, \sigma' \in \mathfrak{S}_{2k}, \forall \eta \in \mathfrak{S}_k$,

$$\mathcal{E}(m_\sigma u_\eta m_{\sigma'}^*) = \begin{cases} c u_{\eta'} & \text{if } \exists \eta' \in \mathfrak{S}_k \ s.t.\ \sigma = (\eta' \sqcup \eta)\sigma' \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

$$\mathcal{E}(m_\sigma^* u_\eta m_{\sigma'}) = \begin{cases} c u_{\eta'} & \text{if } \exists \eta' \in \mathfrak{S}_k \ s.t.\ \sigma = (\eta \sqcup \eta')\sigma' \\ 0 & \text{otherwise,} \end{cases} \tag{2.10}$$

$$\mathcal{E}(m_\sigma u_\eta m_{\sigma'}) = \begin{cases} c' u_{\eta'} & \text{if } \exists \eta' \in \mathfrak{S}_k \ s.t.\ \sigma = \tau(\eta \sqcup \eta')\sigma' \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

$k = 2$



$$= U_{(12)} \cdot \underbrace{M_{(1)(2)(3)(4)}}_{id.}$$

Corollary 1.5. *Let $k \geq 1$ be a fixed integer. Let M_N be a random tensor satisfying Hypothesis 1.2 with parameter (c, c'). We consider the three following random matrices*

$$S_{1,N} = \frac{1}{\sqrt{(2k)!k!c}} \sum_{\sigma \in \mathfrak{S}_{2k}} M_{N,\sigma}, \quad S_{2,N} = \frac{1}{\sqrt{(2k)!k!c}} \sum_{\sigma \in \mathfrak{S}_{2k}} \mathrm{sg}(\sigma)\, M_{N,\sigma},$$

$$S_{3,N} = \frac{1}{2\sqrt{(2k)!k!(c + \Re c')}} \sum_{\sigma \in \mathfrak{S}_{2k}} \left(M_{N,\sigma} + M^*_{N,\sigma}\right),$$

where $\mathrm{sg}(\sigma)$ is the signature of the permutation σ. We denote by δ_0 the Dirac mass at zero, by MP the Marchenko-Pastur distribution of shape parameter $\lambda = 1$ and variance 1, and by SC the standard semicircular distribution.

1. *The empirical eigenvalues distribution of $S_{1,N}S^*_{1,N}$ and $S_{2,N}S^*_{2,N}$ converge to the distribution $(1 - k!^{-1})\delta_0 + k!^{-1}\mathrm{MP}$.*

2. *The empirical eigenvalues distribution of $S_{3,N}$ converges to the distribution $(1 - k!^{-1})\delta_0 + k!^{-1}\mathrm{SC}$.*

Definition 2.2. 1. *For any $\eta \in \mathfrak{S}_k$, let $U_{N,\eta}$ be the unitary matrix of size N^k whose action on a simple tensor $v_1 \otimes \cdots \otimes v_k \in (\mathbb{C}^N)^{\otimes k}$ is*

$$U_{N,\eta}(v_1 \otimes \cdots \otimes v_k) := v_{\eta(1)} \otimes \cdots \otimes v_{\eta(k)}.$$

We denote by $\mathbb{C}\mathfrak{S}_{N,k}$ the vector space spanned by all the matrices $U_{N,\eta}$ for η in $\mathfrak{S}_k$.

2. *Recalling the notation $\Phi_N = \mathbb{E}\big[\frac{1}{N}\mathrm{Tr}\ \cdot\ \big]$, let $\mathcal{E}_N$ be the linear map defined, for any random matrix $A_N \in \mathrm{M}_{N^k}(\mathbb{C})$, by*

$$\mathcal{E}_N(A_N) = \sum_{\eta \in \mathfrak{S}_k} \Phi_N\big[A_N U_{N,\eta}^*\big] U_{N,\eta} \in \mathbb{C}\mathfrak{S}_{N,k}.$$

3. Convergence in distribution and circular families

Definition 2.2. 1. *For any $\eta \in \mathfrak{S}_k$, let $U_{N,\eta}$ be the unitary matrix of size N^k whose action on a simple tensor $v_1 \otimes \cdots \otimes v_k \in (\mathbb{C}^N)^{\otimes k}$ is*

$$U_{N,\eta}(v_1 \otimes \cdots \otimes v_k) := v_{\eta(1)} \otimes \cdots \otimes v_{\eta(k)}.$$

We denote by $\mathbb{C}\mathfrak{S}_{N,k}$ the vector space spanned by all the matrices $U_{N,\eta}$ for η in $\mathfrak{S}_k$.

2. *Recalling the notation $\Phi_N = \mathbb{E}\big[\frac{1}{N}\mathrm{Tr}\ \cdot\ \big]$, let $\mathcal{E}_N$ be the linear map defined, for any random matrix $A_N \in \mathrm{M}_{N^k}(\mathbb{C})$, by*

$$\mathcal{E}_N(A_N) = \sum_{\eta \in \mathfrak{S}_k} \Phi_N\big[A_N U_{N,\eta}^*\big] U_{N,\eta} \in \mathbb{C}\mathfrak{S}_{N,k}.$$

Definition-Proposition 2.9. *Given any sequence* $(\mathcal{M}_n)_{n\geq 1}$ *of linear maps* $\mathcal{M}_n : \mathcal{A}^n \to \mathbb{C}\mathfrak{S}_k$*, we define canonically the collection* $(\mathcal{M}_\xi)_{\xi\in\mathrm{NC}}$ *as follows. Let* ξ *be a non-crossing partition of* $[n]$*. There always exists an interval block* $B = \{i, i+1, \dots, i+n'-1\}$ *of* ξ*, for some* $i \in [n]$ *and* $n' \in [n-i+1]$*. Assume that* B *is the interval block of smallest indices, namely* $i = \min\big\{j \in [n] \,|\, \exists\, m' \in [n] \text{ s.t. } \{j, j+1 \cdots, j+m'-1\} \in \xi\big\}$*. Then for all* $a_1, \dots, a_n$ *in* $\mathcal{A}$*, we set, inductively on* n*,*

$$\begin{aligned}&\mathcal{M}_\xi(a_1, \dots, a_n)\\ &\quad = \mathcal{M}_{\xi\setminus B}\big(a_1, \dots, a_{i-1}\mathcal{M}_{n'}(a_i, a_{i+1}, \dots, a_{i+n'-1}), a_{i+n'}, \dots, a_n\big),\end{aligned}$$

where $\xi \setminus B \in \mathrm{NC}(n-n')$ *is obtained by removing the block* B *and shifting the indices greater than* $i+n'-1$*, with the convention* $\mathcal{M}_{\{\emptyset\}} = \mathrm{id}_{\mathbb{C}\mathfrak{S}_k}$*. This relation entirely characterizes the collection* $(\mathcal{M}_\xi)_{\xi\in\mathrm{NC}}$ *in terms of* $(\mathcal{M}_n)_{n\geq 1}$*.*

Remark 2.10. *For* $k = 1$*, one has* $\mathbb{C}\mathfrak{S}_1 = \mathbb{C}$ *and the functions* $\mathcal{M}_\xi$ *satisfy a trivial factorization,*

$$\mathcal{M}_\xi(a_1, \dots, a_n) = \prod_{B=\{i_1<\cdots<i_{n'}\}\in\xi} \mathcal{M}_{n'}(a_{i_1}, \dots, a_{i_{n'}}). \tag{2.6}$$

Formula (2.6) *is not valid for* $k \geq 2$ *if the maps* $\mathcal{M}_\xi$ *are not* $\mathbb{C}\mathfrak{S}_k$*-linear, e.g.* $\mathcal{M}_2(ab, a') \neq \mathcal{M}_2(a, a')b$ *for some* $a, a' \in \mathcal{A}$*,* $b \in \mathbb{C}\mathfrak{S}_k$*. The general definition of* $\mathcal{M}_\xi$ *depends on the nesting structure of the blocks of* ξ*.*

Definition-Proposition 2.11. *The* $\mathfrak{S}_k$*-free cumulants on* $(\mathcal{A}, \mathbb{C}\mathfrak{S}_k, \mathcal{E})$ *are the unique collection* $(\mathcal{K}_n)_{n\geq 1}$ *of linear maps* $\mathcal{K}_n : \mathcal{A}^n \to \mathbb{C}\mathfrak{S}_k$ *such that, for all* $n \geq 1$ *and* $a_1, \dots, a_n$ *in* $\mathcal{A}$*,*

$$\mathcal{E}(a_1 \cdots a_n) = \sum_{\xi\in\mathrm{NC}(n)} \mathcal{K}_\xi(a_1, \dots, a_n).$$

Each $\mathcal{K}_\xi$ *is a* $\mathbb{C}\mathfrak{S}_k$*-module map, that is*

$$\begin{aligned}\mathcal{K}_\xi(a_1, \dots, a_{i-1}, a_i b, a_{i+1}, \dots, a_n) &= \mathcal{K}_\xi(a_1, \dots, a_{i-1}, a_i, b a_{i+1}, \dots, a_n)\\ \mathcal{K}_\xi(b a_1, a_2, a_3, \dots, a_{n-1}, a_n b') &= b\,\mathcal{K}_\xi(a_1, \dots, a_n) b',\end{aligned}$$

Operator-valued free cumulants

Definition 2.12. *Let $(\mathcal{A}, \mathbb{C}\mathfrak{S}_k, \mathcal{E})$ be a $\mathfrak{S}_k^*$-probability space.*

1. *A collection $\mathbf{m} = (m_j)_{j \in J}$ of elements in $\mathcal{A}$ is* <u>*$\mathfrak{S}_k$-circular*</u> *whenever the following $\mathfrak{S}_k$-cumulants of order greater than two vanish:*

$$\mathcal{K}_n(m_{j_1}^{\varepsilon_1} b_1, m_{j_2}^{\varepsilon_2} b_2, \cdots, m_{j_L}^{\varepsilon_L} b_L) = 0,$$

for all $L \geq 3$ and all $j_\ell \in J, \varepsilon_\ell \in \{1, *\}, b_\ell \in \mathbb{C}\mathfrak{S}_k$ with $\ell \in [L]$.

2. *Let $A_1, \ldots, A_n$ be ensembles of elements of $\mathcal{A}$. The A_i's are* <u>*free over $\mathfrak{S}_k$*</u> *(or $\mathfrak{S}_k$-free) if and only if the mixed $\mathfrak{S}_k$-cumulants vanish:*

$$\mathcal{K}_n(m_1^{\epsilon_1} b_1, m_2^{\epsilon_2} b_2, \ldots, m_L^{\epsilon_L} b_L) = 0,$$

for all $\varepsilon_\ell \in \{1, *\}, b_\ell \in \mathbb{C}\mathfrak{S}_k, \ell \in [L]$, and for all $m_1, \ldots, m_L$ in the union $A_1 \cup \cdots \cup A_n$ but not all in a single set A_i (i.e. $\exists \ell \neq \ell'$ such that $m_\ell \in A_i, m_{\ell'} \in A_{i'}$ and $i \neq i'$).

Circular families

Main intuition: if $X_1^{(N)}, \ldots, X_r^{(N)} \in \mathcal{M}_N(\mathbb{C})$ are iid Ginibre random matrices, then

iid $\mathcal{N}_{\mathbb{C}}(0,1)$ entries

$$\left(X_1^{(N)}, \ldots, X_r^{(N)}\right) \xrightarrow[N \to \infty]{\text{distr.}} \underbrace{(c_1, \ldots, c_r)}_{\text{free circulant element}}$$

A standard circulant element:

$$\varphi(c) = \varphi(c^*) = 0$$

$$\varphi(c^2) = \varphi((c^*)^2) = 0 \qquad \varphi(cc^*) = \varphi(c^*c) = 1$$

$$K_p(c^{\varepsilon_1}, c^{\varepsilon_2}, \ldots, c^{\varepsilon_p}) = 0 \quad \forall\, p \geq 3 \text{ and } \varepsilon_i = 1, *$$

Scalar-free sub-families (generalizing the Mingo-Popa transpose freeness result)

Corollary 2.17. *Let M_N be a random tensor satisfying Hypothesis 1.2 with parameter (c, c').*

there are $\binom{2k}{k}$ such cosets

1. *If $c' = 0$, then the flattenings of M_N indexed by different $\mathfrak{S}_{k,k}$-cosets are asymptotically $\mathfrak{S}_k$-free.* (and also $\mathbb{C}$-free)
2. *If $c' \neq 0$, then the flattenings of M_N indexed by different $\mathfrak{S}_{k,k} \rtimes \langle \tau \rangle$-cosets are asymptotically $\mathfrak{S}_k$-free.*

In each case, different flattenings belonging to the same coset are not $\mathfrak{S}_k$-free.