Equivalent definitions for quantum channels

#lectures

Theorem. Let $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a linear map. Then, the following are equivalent:

1. The map Φ is a quantum channel, i.e. Φ is *completely positive* and *trace preserving*
2. The *Choi matrix* C_Φ is positive semidefinite and normalised

$$C_\Phi \geq 0 \text{ and } \operatorname{Tr}_B C_\Phi = I_A$$

3. *Kraus decomposition*: there exist operators

$$K_1, \ldots, K_r : \mathcal{H}_A \to \mathcal{H}_B$$

such that

$$\Phi(X) = \sum_{i=1}^{r} K_i X K_i^*$$

and

$$\sum_{i=1}^{r} K_i^* K_i = I_A$$

4. *Stinespring dilation*: there exist an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(X) = \operatorname{Tr}_E(VXV^*)$$

Proof. We shall prove 4 implications.

$(1) \implies (2)$: follows easily from

$$C_\Phi = \dim \mathcal{H}_A \cdot [\mathrm{id}_{A'} \otimes \Phi](\omega_{A'A})$$

$(2) \implies (3)$: diagonalise the Choi matrix

$$C_\Phi = \sum_{i=1}^{r} \lambda_i |z_i\rangle\langle z_i|,$$

where $r = \operatorname{rank} C_\Phi$ and $|z_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ are its eigenvectors. Let $Z_i : \mathcal{H}_A \to \mathcal{H}_B$ be the corresponding operators, and set $K_i := \sqrt{\lambda_i} Z_i$. We have, for arbitrary $x \in \mathcal{H}_A$

and $y \in \mathcal{H}_B$,

$$\begin{aligned}
\langle y|\Phi(|x\rangle\langle x|)|y\rangle &= \langle x \otimes y|C_\Phi|x \otimes y\rangle \\
&= \sum_i \lambda_i \langle x \otimes y|z_i\rangle \langle z_i|x \otimes y\rangle \\
&= \sum_i \lambda_i \langle y|Z_i|x\rangle \langle x|Z_i^*|y\rangle \\
&= \langle y\Big| \sum_i K_i|x\rangle\langle x|K_i^* \Big|y\rangle
\end{aligned}$$

$(3) \implies (4)$: take $\mathcal{H}_E := \mathbb{C}^r$ and define

$$Vx := \sum_{i=1}^{r} K_i x \otimes |i\rangle$$

We can check

$$V|x\rangle\langle x|V^* = \sum_{i,j} K_i|x\rangle\langle x|K_j^* \otimes |i\rangle\langle j|$$

hence

$$\mathrm{Tr}_E(V|x\rangle\langle x|V^*) = \sum_i K_i|x\rangle\langle x|K_i^* = \Phi(|x\rangle\langle x|)$$

Moreover, V is an isometry

$$V^*V = \sum_{i,j} K_i^* K_j \langle i|j\rangle = \sum_i K_i^* K_i = I_A$$

$(4) \implies (1)$: start by checking trace preservation:

$$\mathrm{Tr}\,\mathrm{Tr}_E(VXV^*) = \mathrm{Tr}(VXV^*) = \mathrm{Tr}(V^*VX) = \mathrm{Tr}\,X$$

For complete positivity, consider an extra system $\mathcal{H}_R$ and a positive semidefinite operator $Z \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_R)$. We have

$$[\Phi \otimes \mathrm{id}_R](Z) = \mathrm{Tr}_E\left((V \otimes I_R)Z(V \otimes I_R)^*\right) \geq 0$$