① Classical and Q. Inf. Th.

Sets vs Hilbert spaces

classical Inf Th.

Finite alphabet

$\Sigma_A = \{1, 2, \dots, d\}$

different letters

$i \neq j$

Quantum Inf Th

fin dim, $\mathbb{C}$- Hilbert spaces

$\mathcal{H}_A = \mathbb{C}^d =$

span $\{|1\rangle, |2\rangle, \dots, |d\rangle\}$

$|x\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$

superposition

orthogonal vectors

$|x\rangle \perp |y\rangle$

States

Classical

probability measures supported on Σ_A

$\mathcal{P}(\Sigma_A)$

$p \in \mathbb{R}^d \quad p_i \geq 0$

$\sum p_i = 1$

$p \longrightarrow \operatorname{diag}(p)$

extreme points

$\{\delta_i\}_{i \in \Sigma_A}$

Quantum

density matrices

$\rho \in \mathcal{B}(\mathcal{H}_A) \simeq \mathcal{M}_d(\mathbb{C})$ so

ρ is positive semidefinite

$\rho \geq 0$

$\operatorname{Tr} \rho = 1 \qquad \rho \in \mathcal{D}(\mathcal{H}_A)$

$\operatorname{diag}(p)$

ext $\mathcal{D}(\mathcal{H}_A)$ = rank-1 proj

$\{|x\rangle\langle x| : \|x\| = 1 \ x \in \mathbb{C}^d\}$

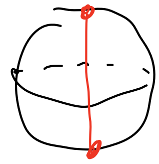
Example $d=2$ bits

$p=(1-t)\delta_0+t\delta_1$

$\mathcal{P}(\Sigma_2)=$ segment

$d=2$ qubits

$\mathcal{D}(\mathbb{C}^2)=$ Bloch ball

$=$ 

$$\rho=\frac{1}{2}(I+x\cdot\sigma_x+y\,\sigma_y+z\,\sigma_z)$$

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\quad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\rho\geq 0 \iff \|(x,y,z)\|\leq 1$$

center of the Bloch ball $= I_2/2$

uniform distributions

$p=\left(\frac{1}{d},\frac{1}{d},\dots,\frac{1}{d}\right)$

$\rho_*=I_d/d$

maximally mixed state

Bipartite and multipartite systems

Classical

Quantum

joint system A+B

$\times$

cartesian product

$\Sigma_{AB}=\Sigma_A\times\Sigma_B$

$\{(i,j): i\in\Sigma_A,\ j\in\Sigma_B\}$

$\otimes$

tensor product

$\mathcal{H}_{AB}=\mathcal{H}_A\otimes\mathcal{H}_B$

$\mathcal{D}(\mathcal{H}_{AB})$ $\rho_{AB}\geq 0$

$\mathrm{Tr}\,\rho_{AB}=1$

$p_{AB}(i,j) \geq 0$

$\sum_{ij} p_{AB}(i,j) = 1$

$\mathcal{H}_{AB} = \mathrm{span}\{ |x\rangle \otimes |y\rangle : x \in \mathcal{H}_A, y \in \mathcal{H}_B \}$

product measures

$p_{AB} = p_A \times p_B$

$p_{AB}(i,j) = p_A(i)\, p_B(j)$

(independent r.v.)

product states

$\rho_{AB} = \rho_A \otimes \rho_B$

$$X \otimes Y = \begin{bmatrix} x_{11} Y & x_{12} Y & \\ & & \ddots \\ & & & x_{mm} Y \end{bmatrix}$$

every bipartite state is a convex combination of product states.

$p_{AB} = \sum_{ij} p_{AB}(i,j) \cdot \delta_i \times \delta_j$

not allways the case !

there exist entangled states

convex comb. of prod. st.

$SEP(A,B) := \mathrm{Conv}\{ \rho_A \otimes \rho_B : \rho_A \in D(\mathcal{H}_A), \rho_B \in D(\mathcal{H}_B) \}$

$SEP \ni \rho = \sum t_i \, \rho_A^{(i)} \otimes \rho_B^{(i)}$

$t_i \geq 0 \quad \sum t_i = 1$

$\rho_{A,B}^{(i)} \in D(\mathcal{H}_{A,B})$

extreme separable states

$\mathrm{ext}(SEP(A,B)) = \{ |x\rangle\langle x| \otimes |y\rangle\langle y| : x \in \mathcal{H}_A, y \in \mathcal{H}_B \}$

a pure state $z \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$

$z \in SEP \iff z = x \otimes y$

pure sep states in $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$z = |1\rangle \otimes |1\rangle, \quad \frac{1}{\sqrt{2}}\left(|1\rangle \otimes |1\rangle + |1\rangle \otimes |2\rangle\right)$$

$$= |1\rangle \otimes \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$$

entangled pure states.

$$z = \frac{1}{\sqrt{2}}\left(|1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle\right) \neq |x\rangle \otimes |y\rangle$$

In general maximally entangled state

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$$

$$\omega = |\Omega\rangle\langle\Omega| \in \mathcal{D}(\mathbb{C}^d \otimes \mathbb{C}^d)$$

$d=2$

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

classical

marginalization

$AB \to A$

$$p_A(i) = \sum_j p_{AB}(ij)$$

quantum

partial trace

$\rho_{AB} \to \rho_A$

$$\rho_A = [\mathrm{id}_A \otimes \mathrm{Tr}_B](\rho_{AB})$$

$$= \mathrm{Tr}_B(\rho_{AB})$$

<u>example</u> $\mathrm{Tr}_A\, \omega = \mathrm{Tr}_B\, \omega = \mathrm{Id}/d$

Quantum measurements

Classical

$\mathbb{P}(i) = p_A(i)$

quantum $\rho \in D(\mathcal{H}_A)$

POVM positive operator valued measure

$M = \{M_x\}_{x \in \Sigma}$

$M_x \in B(\mathcal{H}_A)$ $M_x \geq 0$

$\sum M_x = I_A$

Born rule

$$\underset{\Sigma}{\mathbb{P}(\underset{\Uparrow}{x})} = Tr(M_x \cdot \rho)$$

$$= \langle M_x, \rho \rangle$$

<u>Examples</u>

- basis measurements

  $\Sigma = [d] = \{1, \dots, d\}$

  $M_x = |x\rangle\langle x|$

  $\mathbb{P}(x) = \langle x | \rho | x \rangle$

- trivial measurements

  $M_x = q(x) \cdot I_A$

  $\mathbb{P}(x) = q(x) \cdot \underbrace{Tr(I_A \cdot \rho)}_{=1}$

State discrimination

Alice can prepare ρ_0 or ρ_1

w. prob p_0 p_1

$p_0 + p_1 = 1$

Bob receives ρ

has to decide

if $\rho = \rho_0$

or $\rho = \rho_1$

Win if Bob guesses correctly.

Want : $\mathbb{P}(\text{win}) = p_0 \, \mathbb{P}(\text{Bob answers } 0 \text{ given } \rho = \rho_0) + p_1 \cdots$

Bob can measure ρ using $M = (M_0, M_1)$

$M_{0,1} \geq 0 \qquad M_0 + M_1 = I_d.$

$$\mathbb{P}(\text{win}) = p_0 \operatorname{Tr}(M_0 \cdot \rho_0) + p_1 \operatorname{Tr}(M_1 \cdot \rho_1)$$

Thm [Holevo, Helstrom] $p_0 = p_1 = 1/2$

$$\max_{M_{0,1}} \mathbb{P}(\text{win}) = \frac{1}{2} + \frac{1}{4} \|\rho_0 - \rho_1\|_1$$

where $\| \; \|_1$ = trace norm
nuclear norm
Schatten-1 norm

$$\|X\|_1 = \operatorname{Tr}|X| = \operatorname{Tr}\left[(X^* X)^{1/2}\right]$$

$\|p - q\|_{TV} \hookrightarrow \|\rho_0 - \rho_1\|_1$

$\phantom{\|p-q\|_{TV}} = \sum_i |p(i) - q(i)|$

if $\rho_0 = \operatorname{diag}(p)$
$\rho_1 = \operatorname{diag}(q)$

$\max \mathbb{P}(\text{win}) = \frac{1}{2} + \frac{1}{2} \|p - q\|_{TV}$

sketch of proof. diag

$$\max_{\substack{M_0, M_1 \geq 0 \\ M_0 + M_1 = I}} \underset{= 1/2}{p_0} \langle \underset{\text{diag}}{M_0}, \underset{\text{diag}}{\rho_0} \rangle + \underset{= 1/2}{p_1} \langle \underset{\text{diag}}{M_1}, \underset{\text{diag}}{\rho_1} \rangle$$

$$\max_{0 \leq M_0 \leq I} \frac{1}{2}\langle M_0, \rho_0 - \rho_1 \rangle + \underbrace{\frac{1}{2}\langle I, \rho_1 \rangle}_{= 1/2}$$

$$\operatorname{argmax} = \mathbb{1}(\rho_0 - \rho_1)_+$$

if $\rho_0 - \rho_1 = \sum \lambda_i |x_i\rangle\langle x_i|$

$$M = \sum \mathbb{1}_{\lambda_i \geq 0} |x_i\rangle\langle x_i|$$

$$P_{max} = \frac{1}{2} + \frac{1}{2} \sum \lambda_i \cdot \mathbb{1}_{\lambda_i \geq 0}$$

$$|\rho_0 - \rho_1| = \sum |\lambda_i| \; |x_i\rangle\langle x_i|$$

$$\|\rho_0 - \rho_1\|_1 = \mathrm{Tr}\, |\rho_0 - \rho_1| = \sum |\lambda_i| = \sum_{\lambda_i \geq 0} \lambda_i + \sum_{\lambda_i < 0} (-\lambda_i)$$

$$\stackrel{!}{=} 2 \sum_{\lambda_i \geq 0} \lambda_i$$

since $\mathrm{Tr}(\rho_0 - \rho_1) = 0 = \sum \lambda_i$

② Quantum Channels

Classical

cl. channel: transformations of classical states i.e. prob. meas.

$$N : \mathbb{R}^{|\Sigma_A|} \to \mathbb{R}^{|\Sigma_B|}$$

$\forall a \in \Sigma_a$
$b \in \Sigma_b$

$$N_{b|a} = \langle \delta_b, N(\delta_a) \rangle \geq 0$$

$$\xrightarrow{a} \boxed{N} \xrightarrow{b}$$

$$\forall a \quad \sum_b N_{b|a} = 1$$

Markov kernels
Conditional prob., ...
Stochastic mat.

Quantum

q. channels = transformations of q. states (density matrices)

$$\Phi : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$$

linear

$\to$ Φ is completely positive CP

$\forall \mathcal{H}_R$ $\quad \Phi \otimes id_R$ positive

$Z_{AR} \geq 0 \quad (\Phi \otimes id_R)(Z) \geq 0$

$\to$ Φ is trace preserving TP

$\forall X \quad \mathrm{Tr}\,\Phi(X) = \mathrm{Tr}\,X$

Why $\underline{\text{complete}}$ positivity ?
(automatic in the classical case)

$\exists$ positive maps which are not completely positive e.g. the transposition

$$\begin{aligned} \mathrm{transp} : M_2 &\longrightarrow M_2 \\ X &\longmapsto X^T \end{aligned} \quad \text{is positive}$$

$$(\mathrm{transp} \otimes id_2)(\omega_2) \ngeq 0$$

Choi matrix

$\Phi : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ linear

$$\to \quad \underset{\cap \; B(\mathcal{H}_A \otimes \mathcal{H}_B)}{C_\Phi} := \sum_{i,j=1}^{\dim \mathcal{H}_A} \underbrace{|i\rangle\langle j|}_{E_{ij}} \otimes \Phi(|i\rangle\langle j|)$$

Choi's theorem :

Φ is CP $\iff C_\Phi \geq 0$

Φ is TP $\iff \mathrm{Tr}_B C_\Phi = I_A$

Examples

Classical

$$N_{b|a} = \frac{1}{|\Sigma_B|}$$

$$N_{b|a} = \mathbb{1}_{b=\pi(a)}$$

$\Sigma_A = \Sigma_B$, $\pi \in S_\Sigma$ permutation

Quantum

depolarizing channel

$$\Delta(X) = \mathrm{Tr}(X) \frac{I_B}{\dim \mathcal{H}_B}$$

unitary channels

$$\Phi_U(X) = UXU^*$$

for $U \in \mathcal{U}_{\mathcal{H}_A = \mathcal{H}_B}$

measurement channels

$M = (M_x)_{x=1}^{k}$ a POVM

$(M_x \geq 0 \quad \sum_x M_x = I)$

$$\Phi_M(\rho) = \sum_{x=1}^{k} \langle M_x, \rho \rangle \cdot |x\rangle\langle x|$$

Kraus decomposition $\quad \Phi$ is CP & TP $\Longleftrightarrow$

$\exists\ K_1, \dots, K_r : \mathcal{H}_A \to \mathcal{H}_B$ s.t.

$$\Phi(X) = \sum_{i=1}^{r} K_i X K_i^*$$

$$\sum_{i=1}^{r} K_i^* K_i = I_A$$

Stinespring dilation $\quad \Phi$ is CP & TP $\Longleftrightarrow$

$\exists\ V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ isometry

s.t. $\Phi(X) = \mathrm{Tr}_E(V X V^*)$

Kraus $\downarrow$ Stinespring

$$V x = \sum_{i=1}^{r} K_i x \otimes |i\rangle$$

$$\mathcal{H}_E = \mathrm{span}\,\{|1\rangle, |2\rangle, \dots, |r\rangle\}$$

$$\min r = \mathrm{rank}\, C_\Phi$$

N Markov matrix $\rightsquigarrow \Phi^N$

$$\Phi^N(X) = \sum_{\substack{a \in \Sigma_A \\ b \in \Sigma_B}} N_{b|a} \langle a|X|a\rangle\, |b\rangle\langle b|$$

$$C_{\Phi^N} = \sum N_{b|a}\, |a\rangle\langle a| \otimes |b\rangle\langle b|$$

Different notions of positivity for quantum maps

Map $\Phi : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ Choi matrix $C_\Phi \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$

positive maps

$X \geq 0 \Rightarrow \Phi(X) \geq 0$

block-positive matrix

$\langle x \otimes y | C_\Phi | x \otimes y \rangle \geq 0$

$\forall x \in \mathcal{H}_A \quad \forall y \in \mathcal{H}_B$

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completely positive maps

$\forall \mathcal{H}_R \quad Z_{AR} \geq 0 \Rightarrow [\Phi \otimes id_R](Z) \geq 0$

positive semidef.

$\forall z \in \mathcal{H}_A \otimes \mathcal{H}_B$

$\langle z | C_\Phi | z \rangle \geq 0$

dual

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entanglement breaking

(super-positive)

$\forall \mathcal{H}_R \quad Z_{AR} \geq 0 \Rightarrow (\underset{A\to B}{\Phi} \otimes id_R)(\underset{AR}{Z}) \in SEP(B, R)$

$C_\Phi \in SEP(A, B)$

③ Classical capacity of quantum channels

Classical Shannon

$$H(p) = -\sum_{i \in \Sigma} p_i \log p_i$$

$$H(p) \in [0, \log |\Sigma|]$$

$p = \delta_i$ $\quad$ $p = \left(\frac{1}{|\Sigma|}, \dots, \frac{1}{|\Sigma|}\right)$

$H(A)_p$

Quantum von Neumann

$$S(\rho) = -\operatorname{Tr} \rho \log \rho$$
$$= H(\lambda_\rho)$$

spectrum of ρ

$$S(\rho) \in [0, \log \dim \mathcal{H}]$$

$\rho = |x\rangle\langle x|$ pure state $\quad$ $\rho = \frac{I}{\dim \mathcal{H}}$

Conditional entropy

$$H(A|B)_p = H(AB)_p - H(B)_p \geq 0$$

$$S(A|B)_\rho = S(AB)_\rho - S(B)_\rho$$
$$= S(\rho_{AB}) - S(\rho_B)$$

can be negative!

$\rho_{AB} =$ max. ent. state $= \omega$
$= |\Omega\rangle\langle\Omega|$

$$|\Omega\rangle = \frac{1}{d} \sum |i\rangle \otimes |i\rangle$$

$$S(\rho_{AB}) = 0$$

$$S\left(\rho_B = I/d\right) = \log d$$

relative entropy

$$D_{KL}(p\|q) = \sum p_i \log \frac{p_i}{q_i} \qquad D(\rho\|\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$$

$$D(\rho\|\sigma) \geq 0$$

Data-processing inequality

$\forall\, \Phi$ q. channel

$$D(\rho\|\sigma) \geq D(\Phi(\rho)\,\|\,\Phi(\sigma))$$

Mutual information

$$I(A:B)_p = H(A)_p + H(B)_p - H(AB)_p = H(B)_p - H(B|A)_p$$

$$I(A:B)_\rho = S(A)_\rho + S(B)_\rho - S(AB)_\rho$$

$I(A:B) \geq 0 \qquad = 0$ iff $\quad p_{AB} = p_A \times p_B$

$\rho_{AB} = \rho_A \otimes \rho_B$

Shannon's channel coding theorem

N is a classical channel $\qquad N = (N_{b|a})$

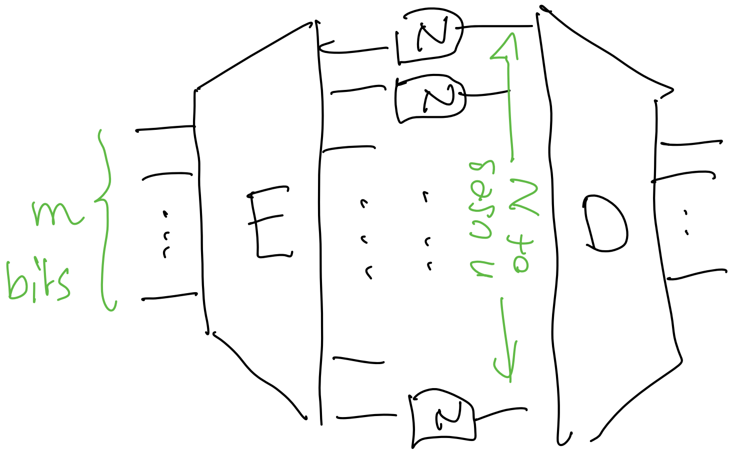
$$a \longrightarrow \boxed{N} \longrightarrow b \qquad N : \Sigma_A \to \Sigma_B$$

coding scheme (m, n, δ) for N is

a pair $\quad E : \{0,1\}^m \longrightarrow \Sigma_A^n$

$D : \Sigma_B^n \longrightarrow \{0,1\}^m \quad$ such that

$\forall i \in \{0,1\}^m \quad \mathbb{P}\left(D \circ N^{n} \circ E(i) = i \right) \geq 1-\delta$



achievable rate R

R is achievable if $\forall n\ \exists\ (m_n, n, \delta_n)$ coding scheme for N

s.t. $\dfrac{m_n}{n} \to R \quad \delta_n \to 0$

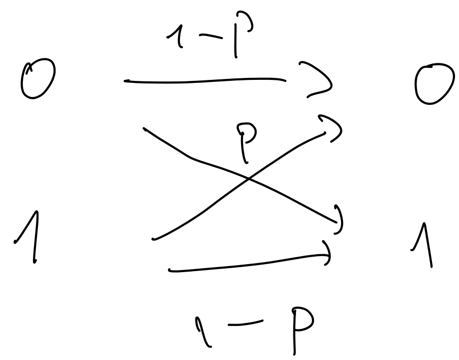
Capacity $C(N) = \sup\{R : R \text{ is achievable}\}$

<u>Theorem</u> $\quad C(N) = \sup_{p_A} I(A:B)_{p^N_{AB}}$

$$p^N_{AB}(a,b) := p_A(a)\, N_{b|a}$$

<u>Example</u> Binary symmetric channel.

$\Sigma_A = \Sigma_B = \{0,1\}$



BSC_p

$C(BSC_p) = 1 - H((p, 1-p))$

Classical capacity of a quantum channel

coding scheme for $\Phi : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$

(m, n, δ)

$E : \{0,1\}^m \to B(\mathcal{H}_A^{\otimes n})$

$D : B(\mathcal{H}_B^{\otimes n}) \to \{0,1\}^m$ POVM

$\forall\, i \in \{0,1\}^m \quad \langle D_i , \Phi^{\otimes n}[E(i)] \rangle \geq 1-\delta$

achievable rate : $\forall n\ \exists (m_n, n, \delta_n)$ coding st

$$\frac{m_n}{n} \to R \quad , \quad \delta_n \to 0$$

capacity $\quad C(\Phi) = \max\{R : R \text{ is achievable}\}$

$D : B(\mathcal{H}_B^{\otimes n}) \to \{0,1\}^m \simeq \operatorname{diag} \mathcal{M}_{2^m}$ decoding channel.

$$D(\rho) = \sum_{i\in\{0,1\}^m} \langle D_i , \rho \rangle \cdot |i\rangle\langle i|$$

Theorem (HSW)

$$C(\Phi) = \lim_{n\to\infty} \frac{1}{n} \chi(\Phi^{\otimes n})$$

where

$$\chi(\Psi) = \max_{\{p(i),\, \rho_i\}_{i\in I}} S\Big(\sum_i p_i \Psi(\rho_i)\Big) - \sum_i p_i S(\Psi(\rho_i))$$

Holevo capacity of Ψ

if χ were additive i.e.

$$\chi(\Phi_1 \otimes \Phi_2) = \chi(\Phi_1) + \chi(\Phi_2)$$

then $C(\Phi) = \chi(\Phi)$ No!

<u>Fact</u> [Shor]

χ is additive $\Leftrightarrow$ S_{min} additive

$$S_{min}(\Phi) = \min_{\rho \in D(\mathcal{H}_A)} S(\Phi(\rho)) = \min_{\sigma \in \Phi(D(\mathcal{H}_A))} S(\sigma)$$

$$\chi(\Phi) = \max_{p,\rho} \underbrace{S\Big(\sum p_i \Phi(\rho_i)\Big)}_{\leq \log \dim \mathcal{H}_B} - \sum p_i \underbrace{S(\Phi(\rho_i))}_{\geq S_{min}}$$

$$\leq \log \dim \mathcal{H}_B - S_{min}(\Phi)$$

- if Φ is covariant $\exists$ irreps U_g, V_g
  $\forall X \quad V_g \Phi(X) V_g^\dagger = \Phi(U_g X U_g^\dagger)$
  then $\chi(\Phi) = \log \dim \mathcal{H}_B - S_{min}(\Phi)$

<u>Example</u> depolarizing channel

$$D_\lambda(\rho) = (1-\lambda)\rho + \lambda \, I/d$$

$$\Delta_\lambda : \mathcal{M}_d \longrightarrow \mathcal{M}_d \qquad \mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^d$$

Fact [King) $\forall\, \Phi$

$$\chi(\Delta_\lambda \otimes \Phi) = \chi(\Delta_\lambda) + \chi(\Phi)$$

$$C(\Delta_\lambda) = \chi(\Delta_\lambda) = \log d - S_{min}(\Delta_\lambda)$$

$$= \log d - H\Big(1-\lambda+\frac{\lambda}{d}, \underbrace{\frac{\lambda}{d}, \ldots, \frac{\lambda}{d}}_{d-1 \text{ times}}\Big)$$

Relation between Shannon's and HSW theorems

$$\Phi : A \to B$$

$$N^{\Phi}_{b|a} = \langle M_b, \Phi(\rho_a) \rangle$$

Here E prepares ρ_a with prob p_a

D measures (M_b)

$$\underbrace{\xrightarrow{a} \boxed{E} \xrightarrow{\rho_a} \boxed{\Phi} \xrightarrow{\sigma} \boxed{D} \xrightarrow{b}}_{\xrightarrow{a} \boxed{N^{\Phi}} \xrightarrow{b}}$$

$$C(N^{\Phi}) = \max_{p_A} I(A:B)_{p_{AB}^{N^{\Phi}}}$$

$$\max_{\rho_a, M_b} C(N^{\Phi}) = \max_{p_A, \rho_a, M_b} I(A:B)_{p_{AB}}$$

$$p_{AB}(a,b) = p_A(a) \langle M_b, \Phi(\rho_a) \rangle$$

$$= \max_{p_A^{(a)}, \rho_a} \underbrace{\max_{M_b} I(A:B)_{p_{AB}}}_{I_{acc}(\mathcal{E}_{p_A(a), \rho_a})}$$

accessible information

$\mathcal{E}$ = state ensemble

$p_A(a)$, $\Phi(\rho_a)$

$$I_{acc}(\mathcal{E}) = \max_{M_b} I(A:B)_{p_{AB}}$$

$$\leq \chi(\mathcal{E})$$

Holevo

$$\chi(\mathcal{E}) = S\left(\sum p_A(a) \Phi(\rho_a)\right) - \sum_a p_A(a) S(\Phi(\rho_a))$$