Monogamy of highly symmetric states

Ion Nechita (LPT Toulouse)
R. Allerstorfer, M. Christandl, D. Grinko, M. Ozols, D. Rochette, P. Verduyn Lunel
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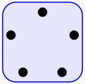
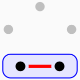
Outline

We introduce the notion of **graph-extendability**

A bipartite symmetric quantum state $\rho =$  is $G =$  -extendible if there exists a global state $\sigma =$  on G such that for all edges $e =$  $\in G$, the reduced state $\sigma_e =$  is equal to ρ.

For given d and n, which *highly symmetric states* (such as Werner, Brauer, and isotropic states) on $\mathbb{C}^d \otimes \mathbb{C}^d$ are G-extendible?

Separability of quantum states

Quantum entanglement

- Quantum states are unit trace positive semidefinite matrices [NC10, Wat18]: $\rho \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})$ such that $\rho \geq 0$, $\operatorname{Tr} \rho = 1$.
- A bipartite quantum state $\rho \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C}) \otimes \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})$ is separable if it can be decomposed as a convex combination of product quantum states:

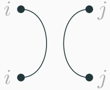
$$\rho = \sum_i \alpha_i \otimes \beta_i \qquad \text{with } \alpha_i, \beta_i \geq 0$$

- A pure (i.e. unit rank) state $\rho = |x\rangle\langle x|$ is separable iff it is product:

$$|x\rangle = |a\rangle \otimes |b\rangle$$

- The maximally entangled state

$$\omega := \frac{1}{d} \sum_{i,j=1}^{d} |ii\rangle\langle jj| = \frac{1}{d}$$



- Deciding whether a given state ρ is separable is an NP-hard problem [Gur03].

Detecting entanglement

- There exist various criteria to detect entanglement or separability

$$\rho \in \mathrm{SEP} \implies \rho^\Gamma := [\mathrm{id} \otimes \mathrm{transp}](\rho) = \sum_i \alpha_i \otimes \beta_i^\top \geq 0$$

$$\left\| \rho - \frac{I}{d} \otimes \frac{I}{d} \right\|_2 \leq \frac{1}{d\sqrt{d^2 - 1}} \implies \rho \in \mathrm{SEP}$$

- The DPS hierarchy [DPS02, DPS04] can certify entanglement using a sequence of semidefinite programs

$$\mathrm{EXT}_k := \left\{ \rho_{AB} \ : \ \exists\, \sigma_{AB_1B_2\cdots B_k} \geq 0 \text{ s.t. } \sigma_{AB_i} = \rho_{AB} \quad \forall i \in [k] \right\}$$

$$\text{all states} = \mathrm{EXT}_1 \supseteq \mathrm{EXT}_2 \supseteq \cdots \supseteq \mathrm{EXT}_k \supseteq \cdots \supseteq \mathrm{EXT}_\infty = \mathrm{SEP}$$

- Easy direction: if ρ is separable, $\rho = \sum_i \alpha_i \otimes \beta_i \rightsquigarrow$ take $\sigma = \sum_i \alpha_i \otimes \beta_i^{\otimes k}$
- Quantitative version [CKMR07]:

$$\rho \in \mathrm{EXT}_k \implies \min_{\sigma \in \mathrm{SEP}} \|\rho - \sigma\|_1 \leq \frac{4d^2}{k}$$

Graph extendability

Monogamy of entanglement

- Monogamy is a fundamental property of quantum entanglement [KW04]. Informally, given 3 quantum parties Alice, Bob, and Charlie:

  Alice cannot be maximally entangled with Bob and Charlie

$$\nexists\, \rho_{ABC} \quad \text{s.t.} \quad \rho_{AB} = \omega \text{ and } \rho_{AC} = \omega$$

- Actually, we have more: given a quantum state ρ_{ABC},

$$\rho_{AB} = \omega \implies \rho_{ABC} = \omega_{AB} \otimes \rho_C$$

- One can give a quantitative statement of monogamy for the case of qubits using the Coffman-Kundu-Wootters monogamy inequality [CKW00]

$$\mathcal{C}^2(A\,|\,BC) \geq \mathcal{C}^2(A\,|\,B) + \mathcal{C}^2(A\,|\,C)$$

  where $\mathcal{C} \in [0,1]$ is the concurrence

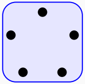
$$\mathcal{C}^2(\psi_{AB}) = 2(1 - \mathrm{Tr}(\rho_A^2)) \quad \text{and} \quad \mathcal{C}(\rho_{AB}) = \min \sum_i p_i \mathcal{C}(\psi_i)$$

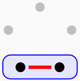
  over all decompositions

$$\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Main definition

A bipartite symmetric quantum state $\rho =$  is $G =$  -extendible if

there exists a global state $\sigma =$  on G such that

for all edges $e =$  $\in G$, the reduced state $\sigma_e =$  is equal to ρ.

- This notion generalizes:

$$n\text{-extendibility} : \exists\, \sigma_{AB_1B_2\cdots B_n} \text{ s.t. } \sigma_{AB_i} = \rho_{AB} \iff K_{1,n}\text{-extendibility}$$

$$n\text{-exchangeability} : \exists\, \sigma_{A_1A_2\cdots A_n} \text{ s.t. } \sigma_{A_iA_j} = \rho_{AB} \iff K_n\text{-extendibility}$$

- Recall: a bipartite symmetric state ρ is called n-exchangeable if there exists a n-partite *symmetric* state σ such that $\rho = \mathrm{Tr}_{n-2}\,\sigma$. The quantum de Finetti theorem [HM76, CFS02, KR05, CKMR07]: ρ is *n-exchangable* $\forall\, n$ iff $\rho = \sum_i \alpha_i \otimes \alpha_i$
- G-extendibility can be formulated as a semidefinite program.

Main result

- Consider isotropic states

$$\rho_I(d) := p\omega + (1-p)\frac{I}{d} \otimes \frac{I}{d}$$

> The largest p for which the isotropic state $\rho_I(d)$ is K_n-extendible is:
>
> $$p_I(n,d) = \begin{cases} \frac{1}{n-1+n \bmod 2} & \text{if } d > n \text{ or either } d \text{ or } n \text{ is even} \\ \min\left\{\frac{2d+1}{2dn+1}, \frac{1}{n-1}\right\} & \text{if } n \geq d \text{ and both } d \text{ and } n \text{ are odd} \end{cases}$$

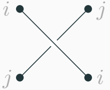
- Compare with optimal p for $K_{1,n}$-extensibility ($\iff$ quantum cloning [KW99])

$$p_I(K_{1,n}, d) = \frac{d+n}{n(d+1)}$$

- Similar results for Werner states and for Brauer states

$$\rho_W(d) := p\frac{\Pi_{\boxminus}}{\operatorname{Tr}\Pi_{\boxminus}} + (1-p)\frac{\Pi_{\boxbar}}{\operatorname{Tr}\Pi_{\boxbar}}, \qquad \rho_B(d) := p\omega + q\frac{\Pi_{\boxminus}}{\operatorname{Tr}\Pi_{\boxminus}} + (1-p-q)\left[\frac{\Pi_{\boxbar}}{\operatorname{Tr}\Pi_{\boxbar}} - \omega\right]$$

$$\Pi_{\boxminus} := \frac{I-F}{2}, \qquad \Pi_{\boxbar} := \frac{I+F}{2}, \qquad F := \sum_{i,j=1}^{d} |ij\rangle\langle ji| =$$



Proof techniques

LB for isotropic states: perfect matchings

- A perfect matching on a graph $G = (V, E)$ is a subset of edges from E, such that every vertex in V is contained in exactly one of those edges.
- There are $(2n-1)!!$ perfect matchings on K_{2n}, and if e is an edge on K_{2n}, then there are $(2n-3)!!$ perfect matchings on K_{2n} containing e.

- Let $E_1, \ldots, E_{(2n-1)!!}$ be all the perfect matchings on K_{2n}, and for each perfect matching E_k, define the quantum state $\rho^{(k)}$ on K_{2n} by

$$\rho^{(k)} := \bigotimes_{e \in E_k} \omega_e \qquad \text{and} \qquad \rho := \frac{1}{(2n-1)!!} \sum_{k=1}^{(2n-1)!!} \rho^{(k)}$$

- For any edge $e \in K_{2n}$, we have

$$\rho_e = \frac{1}{2n-1}\omega + \left(1 - \frac{1}{2n-1}\right)\frac{I}{d^2} \qquad \Longrightarrow \qquad p_I(2n, d) \geq \frac{1}{2n-1}.$$

UB for Werner states: symmetry

- Consider the simpler Werner states $p \cdot \Pi_{\substack{\square\\\square}} / \operatorname{Tr} \Pi_{\substack{\square\\\square}} + (1-p) \cdot \Pi_{\square\square} / \operatorname{Tr} \Pi_{\square\square}$.
- We want to solve, for a graph G with n vertices

$$p_W(G, d) := \max_{\rho, p} p \quad \text{s.t.} \quad \operatorname{Tr}[\Pi_e \rho] = p \quad \forall e \in E, \quad \operatorname{Tr} \rho = 1, \quad \rho \geq 0$$

  where Π_e acts like $\Pi_{\substack{\square\\\square}}$ on the tensor factors associated to the vertices of e and as the identity elsewhere; ρ is a state on $(\mathbb{C}^d)^{\otimes n}$.
- Given an optimal ρ, we can assume wlog that it has symmetry:

$$\forall U \in \mathcal{U}(d) \qquad U^{\otimes n} \rho (U^{\otimes n})^* = \rho$$
$$\forall \pi \in \mathfrak{S}_n \qquad \pi.\rho = \rho$$

  with $\pi.A_1 \otimes A_2 \otimes \cdots \otimes A_n := A_{\pi^{-1}(1)} \otimes Ax_{\pi^{-1}(2)} \otimes \cdots \otimes A_{\pi^{-1}(n)}$.
- By Schur–Weyl duality [Aub18, GO22, Bra37], we have

$$\rho = \sum_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} \beta_\lambda \rho_\lambda$$

  where β_λ is a probability distribution $\{\beta_\lambda : \lambda \vdash n\}$ and ρ_λ are the normalized isotypical projectors.

Representation theory

- The groups $\mathcal{U}(d)$ and $\mathfrak{S}_n$ act on $(\mathbb{C}^d)^{\otimes n}$:

$$U.\,|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle := U\,|x_1\rangle \otimes U\,|x_2\rangle \otimes \cdots \otimes U\,|x_n\rangle$$

$$\pi.\,|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle := \left|x_{\pi^{-1}(1)}\right\rangle \otimes \left|x_{\pi^{-1}(2)}\right\rangle \otimes \cdots \otimes \left|x_{\pi^{-1}(n)}\right\rangle$$

- Schur–Weyl duality: the algebras spanned by the matrices associated to these actions are mutual commutants of each other. Equivalently, the space $(\mathbb{C}^d)^{\otimes n}$ decomposes into isotypic sectors consisting of tensor products of irreps:

$$(\mathbb{C}^d)^{\otimes n} \simeq \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} V_\lambda^{(\mathcal{U})} \otimes V_\lambda^{(\mathfrak{S})}.$$

- Since an optimal ρ commutes is invariant w.r.t. both actions, it must act like the identity on each tensor factor, for every term of the direct sum.
- We have [CKMR07] $\mathrm{Tr}_{[n]\setminus e}\, \rho_\lambda = \alpha^\lambda_{\substack{\square\\\square}} \varepsilon_{\substack{\square\\\square}} + \alpha^\lambda_{\square\square} \varepsilon_{\square\square}$, where

$$\alpha^\lambda_{\substack{\square\\\square}} = \frac{s^*_{\substack{\square\\\square}}(\lambda)}{m_d\left(\substack{\square\\\square}\right) n(n-1)},$$

where $s^*_\mu(\lambda)$ is the shifted Schur function [OO97] and $m_d(\lambda) = \dim V_\lambda^{(\mathcal{U})}$.

Optimization

- Plugging the partial trace expression into the formula for p_W, in the case $G = K_n$, we obtain
$$p_W(\rho) = \sum_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} \beta_\lambda \frac{d(\boxminus) s^*_{\boxminus}(\lambda)}{n(n-1)}$$
- Since β_λ are probability weights, we need to maximize the expression above over partitions $\lambda \vdash n$ with $l(\lambda) \leq d$.
- Using a formula for the shifted Schur function [OO97] we obtain
$$p_W(n,d) = \max_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} \frac{\sum_{d \geq i > j \geq 1} \lambda_i(\lambda_j + 1)}{n(n-1)}$$
- The optimal λ is the tallest approximate rectangle possible, and gives
$$p_W(n,d) = \frac{d-1}{2d} \cdot \frac{(n+k+d)(n-k)}{n(n-1)} + \frac{k(k-1)}{n(n-1)} \quad \text{where } k = n \bmod d$$
- Clearly, if $d \geq n$, $p_W = 1$ is achieved by $\lambda = 1^n$, and ρ is the normalized projection on the anti-symmetric subspace $\Lambda^n(\mathbb{C}^d) \subseteq (\mathbb{C}^d)^{\otimes n}$.

Take home slide

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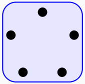
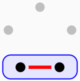
there exists a global state $\sigma =$  on G such that

for all edges $e =$  $\in G$, the reduced state $\sigma_e =$  is equal to ρ.

- For $G = K_{1,n}$ or $G = K_{m,n}$, we obtain the standard DPS hierarchy.
- For all d and n, we compute the range of noise parameters p for which highly symmetric states (Werner, Brauer, isotropic) on $\mathbb{C}^d \otimes \mathbb{C}^d$ are K_n-extendible

$$\rho_I = p \cdot \frac{1}{d} \sum_{ij} |ii\rangle\langle jj| + (1-p) \cdot \frac{I}{d} \otimes \frac{I}{d}$$

- G-extendibility of isotropic states for all n: separability vs. K_n-extendibility

Graph family	Form of ∞-extendible states	Range of p
$K_{1,n}$ or $K_{m,n}$	$\rho = \sum_i \alpha_i \otimes \beta_i$	$\left[\frac{-1}{d^2-1}, \frac{1}{d+1}\right]$
K_n	$\rho = \sum_i \alpha_i \otimes \alpha_i$	$\{0\}$

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